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A WILLIAMS' DECOMPOSITION FOR SPATIALLY DEPENDENT SUPERPROCESSES

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Abstract. We present a genealogy for superprocesses with a non-homogeneous quadratic branching mechanism, relying on a weighted version of the superprocess and a Girsanov theorem. We then decompose this genealogy with respect to the last individual alive (William's decomposition). Letting the extinction time tend to infinity, we get the Q-process by looking at the superprocess from the root, and define another process by looking from the top. Examples including the multitype Feller diffusion and the superdiffusion are provided.

1. Introduction

Even if superprocesses with very general branching mechanisms are known, most of the works devoted to the study of their genealogy are concerned with homogeneous branching mechanisms, that is, populations with identical individuals. Four distinct approaches have been proposed for describing these genealogies. When there is no spatial motion, superprocesses are reduced to continuous state branching processes, whose genealogy can be understood by a flow of subordinators, see Bertoin and Le Gall [5], or by growing discrete trees, see Duquesne and Winkel [13]. With a spatial motion, the description of the genealogy can be done using the lookdown process of Donnelly and Kurtz [11] or the snake process of Le Gall [22]. Some works generalize both constructions to non-homogeneous branching mechanisms: Kurtz and Rodriguez [20] recently extended the lookdown process in this direction whereas Dhersin and Serlet proposed in [10] modifications of the snake.

Using the genealogy, it is natural to consider the corresponding Williams' decomposition, which is named after the work of Williams [32] on the Brownian excursion. After Aldous recognized in [3] the genealogy of a branching process in this excursion, they also designate decompositions of branching processes with respect to their height, see Serlet [31] for the quadratic branching mechanism or Abraham and Delmas [1] for general branching mechanism. Their interest is twice: they allow to understand the behavior of processes at the top, see Goldschmidt and Haas [18] for an application of this approach, and to investigate the process conditioned on non extinction, or Q-process, see [31] and Overbeck [25].

For Markov processes with absorbing states, the Q-process is defined as the process conditioned on non absorption in remote time, see Darroch and Seneta [9]. Lamperti and Ney [21] found a simple construction in the case of discrete branching processes. Later on, Roelly and Rouault [28] provided a superprocess version of this result. Q-processes have intrinsic interest as a model of stochastic population, see Chen and Delmas [7]. They also find application in the study of the associated martingale, see Lyons, Pemantle and Peres [24] in a discrete setting.

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Understanding this martingale allows to better understand the original process, see Engländer and Kyprianou [16] for superprocesses with non homogeneous branching mechanism.

Our primary interest is to present a genealogy for superprocess with a non-homogeneous quadratic branching mechanism, to condition it with respect to its height (this is the William's decomposition), and to study the associated Q-process.

Let $X = (X_t, t \geq 0)$ be an $(\mathcal{L}, \beta, \alpha)$ superprocess over a Polish space E. The underlying spatial motion $Y = (Y_t, t \geq 0)$ is a Markov process with infinitesimal generator $\mathcal{L}$ started at x under P_x. The non-homogeneous quadratic branching mechanism is denoted by $\psi(x, \lambda) = \beta(x)\lambda + \alpha(x)\lambda^2$, for suitable functions β and α (explicit conditions can be found in Section 2). Let $\mathbb{P}_\nu$ be the distribution of X started from the finite measure ν on E, and $\mathbb{N}_x$ be the corresponding canonical measure of X with initial state x. In particular, the process X under $\mathbb{P}_\nu$ is distributed as $\sum_{i\in\mathcal{I}} X^i$, where $\sum_{i\in\mathcal{I}} \delta_{X^i}(dX)$ is a Poisson Point measure with intensity $\int_{x\in E} \nu(dx)\mathbb{N}_x(dX)$. We define the extinction time of X: $H_{\max} = \inf\{t > 0, X_t = 0\}$, and assume that X suffers almost sure extinction, that is $\mathbb{N}_x [H_{\max} = \infty] = 0$ for all $x \in E$. Using an h-transform from Engländer and Pinsky [17] and a Girsanov transformation from Perkins [26], we provide a genealogical structure for the superprocess X, see Proposition 3.12, by transferring the genealogical structure of an homogeneous superprocess.

We define the function $v_h(x) = \mathbb{N}_x [X_h \neq 0] = \mathbb{N}_x [H_{\max} \geq h]$ and a family of probability measures by setting:

$$\forall\ 0 \leq t < h, \quad \frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\partial_h v_{h-t}(Y_t)}{\partial_h v_h(x)}\, \mathrm{e}^{-\int_0^t ds\ \partial_\lambda \psi(Y_s, v_{h-s}(Y_s))},$$

where $\mathcal{D}_t = \sigma(Y_s, 0 \leq s \leq t)$ is the natural filtration of Y, see Lemma 4.10. Using the genealogical structure of X, we give a decomposition of the superprocess X with respect to an individual chosen at random, also called a Bismut decomposition, in Proposition 4.4. The following Theorem, see Corollary 4.13 for a precise statement and Corollary 4.14 for a statement under $\mathbb{P}_\nu$, gives a Williams' decomposition of X, that is a spine decomposition with respect to its extinction time $H_{\max}$.

Theorem. *(Williams' decomposition under $\mathbb{N}_x$) Assume that the $(\mathcal{L}, \beta, \alpha)$ superdiffusion X suffers almost sure extinction and some regularities on α and β.*

(i) *The distribution of H_{max} under $\mathbb{N}_x$ is characterized by: $\mathbb{N}_x[H_{max} > h] = v_h(x)$.*

(ii) *Conditionally on $\{H_{max} = h_0\}$, the $(\mathcal{L}, \beta, \alpha)$ superdiffusion X under $\mathbb{N}_x$ is distributed as $X^{(h_0)}$ constructed as follows. Let $x \in E$ and $Y_{[0,h_0)}$ be distributed according to $\mathrm{P}^{(h_0)}_x$. Consider the Poisson point measure $\mathcal{N} = \sum_{j\in J} \delta_{(s_j, X^j)}$ on $[0, h_0) \times \Omega$ with intensity:*

$$2\mathbf{1}_{[0,h_0)}(s)ds\ \mathbf{1}_{\{H_{max}(X)<h_0-s\}}\alpha(Y_s)\ \mathbb{N}_{Y_s}[dX].$$

The process $X^{(h_0)} = (X^{(h_0)}_t, t \geq 0)$ is then defined for all $t \geq 0$ by:

$$X^{(h_0)}_t = \sum_{j\in J,\, s_j<t} X^j_{t-s_j}.$$

The proof of this Theorem relies on a William's decomposition of the genealogy of X, see Theorem 4.12. Notice it also implies the existence of a measurable family $(\mathbb{N}^{(h)}_x, h > 0)$ of probabilities such that $\mathbb{N}^{(h)}_x$ is the distribution of X under $\mathbb{N}_x$ conditionally on $\{H_{\max} = h\}$.

We shall from now on consider the case of Y a diffusion on $\mathbb{R}^K$ or a pure jump process on a finite state space. The generalized eigenvalue λ_0 of the operator $\beta - \mathcal{L}$ is defined in Pinsky [27] for diffusion on $\mathbb{R}^d$. For finite state space, it reduces to the Perron Frobenius eigenvalue, see Seneta [30]. In both cases, we have:

$$\lambda_0 = \sup\{\ell \in \mathbb{R}, \exists u \in \mathcal{D}(\mathcal{L}), u > 0 \text{ such that } (\beta - \mathcal{L})u = \ell\, u\}\cdot$$

We assume that the space of positive harmonic functions for $(\beta - \lambda_0) - \mathcal{L}$ is one dimensional, generated by a function ϕ_0. From these assumptions, we have that the space of positive harmonic functions of the adjoint of $(\beta - \lambda_0) - \mathcal{L}$ is one dimensional, and we denote by $\tilde{\phi_0}$ a generator of this space. We also assume that ϕ_0 is bounded from below and above by positive constants and that the operator $(\beta - \lambda_0) - \mathcal{L}$ is product critical, that is $\int_E dx\, \phi_0(x)\, \tilde{\phi_0}(x) < \infty$. Thanks to the product-critical property, the probability measure P^{ϕ_0}, given by:

$$\forall t \geq 0, \quad \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\phi_0(Y_t)}{\phi_0(Y_0)}\, \mathrm{e}^{-\int_0^t ds\ (\beta(Y_s) - \lambda_0)},$$

defines a recurrent Markov process (in the sense given by (70)). Since ϕ_0 is bounded from below and from above by two positive constants, the non-negativity of λ_0 implies the weak convergence of the spine that is the weak convergence of $\mathrm{P}^{(h)}_x$ towards $\mathrm{P}^{(\infty)}_x$ which is given by $\mathrm{P}^{\phi_0}_x$, see Proposition 6.8. An explicit expression is given for $\mathrm{P}^{(h)}_x$ and $\mathrm{P}^{\phi_0}_x$ in Lemmas 7.4 and 7.7. The non-negativity of λ_0 implies the almost sure extinction of X, see Lemma 6.2. Under very general conditions, the weak convergence of the spine implies the convergence of the superprocess (Corollary 5.8) and its genealogy (Theorem 5.5). We can easily state them in the particular case of an underlying motion being a diffusion or a pure jump process on a finite state space. We also see that $\mathbb{N}^{(\infty)}_x$, defined below, is actually the law of the Q-process, defined as the weak limit of the probability measures $\mathbb{N}^{(\geq h)}_x = \mathbb{N}_x[\,\cdot\,|H_{\max} \geq h]$, see also Lemma 5.1.

Theorem (Q-process under $\mathbb{N}_x$)**.** *Assume that $\lambda_0 \geq 0$. Let Y be distributed according to $\mathrm{P}^{\phi_0}_x$, and, conditionally on Y, let $\mathcal{N} = \sum_{j\in\mathcal{I}} \delta_{(s_j, X^j)}$ be a Poisson point measure with intensity:*

$$\mathbf{2}\mathbf{1}_{\mathbb{R}^+}(s) ds\ \alpha(Y_s) \mathbb{N}_{Y_s}[dX].$$

Consider the process $X^{(\infty)} = (X^{(\infty)}_t, t \geq 0)$, which is defined for all $t \geq 0$ by:

$$X^{(\infty)}_t = \sum_{j\in J,\, s_j<t} X^j_{t-s_j},$$

and denote by $\mathbb{N}^{(\infty)}_x$ its distribution. Then, for all $t \geq 0$, the distribution of $(X_s, s \in [0,t])$ under $\mathbb{N}^{(h)}_x$ or $\mathbb{N}^{(\geq h)}_x$ converges weakly to $(X^{(\infty)}_s, s \in [0,t])$.

Notice those results also hold under $\mathbb{P}_\nu$ (see Corollary 5.8). It is interesting to notice that the law of the spine P^{ϕ_0} is quite different from that of the backbone given in [17], see also Remark 7.8.

Remark 1.1. As noticed by Li [23], the multitype Dawson Watanabe superprocess can be understood as a single non-homogeneous superprocess on an extended space. The above Theorem on Q-process provides a construction of the Q-process associated to a multitype Dawson Watanabe superprocess considered in Champagnat and Roelly [6], and this construction gives a precise meaning to “the interactive immigration” introduced in Remark 2.8 of [6].

Remark 1.2. In [16], a spinal decomposition of the semi-group of a Doob h-transform of the $(\mathcal{L}, \beta, \alpha)$ superdiffusion is provided, see Theorem 5 of [16]. The second item of Corollary 5.8, together with Lemma 5.1 gives a pathwise decomposition of the genealogy and establishes that the process considered actually is the Q-process.

We also prove the weak convergence of the probability measures $(\mathbb{N}_x^{(h)}, h > 0)$ backward from the extinction time. Let $\mathrm{P}^{(-h)}$ denote the push forward probability measure of $\mathrm{P}^{(h)}$, defined by:

$$\mathrm{P}^{(-h)}((Y_s, s \in [-h, 0]) \in \bullet) = \mathrm{P}^{(h)}((Y_{h+s}, s \in [-h, 0]) \in \bullet).$$

The product criticality assumption yields the existence of a probability measure $\mathrm{P}^{(-\infty)}$ such that for all $x \in E$, $t \geq 0$, and f bounded measurable:

$$\mathrm{E}_x^{(-h)}\left[f(Y_s, s \in [-t, 0])\right] \xrightarrow[h \to +\infty]{} \mathrm{E}^{(-\infty)}\left[f(Y_s, s \in [-t, 0])\right].$$

Once again, the convergence of the spine implies the convergence of the superprocess. The following result corresponds to the second item of Theorem 5.9.

Theorem (Asymptotic distribution at the extinction time)**.** *Assume that $\lambda_0 > 0$. Then the process $(X_{h+s}, s \in [-t, 0])$ under $\mathbb{N}_x^{(h)}$ weakly converges towards $X_{[-t,0]}^{(-\infty)}$, where for $s \leq 0$:*

$$X_s^{(-\infty)} = \sum_{j \in J,\ s_j < s} X_{s-s_j}^j,$$

and conditionally on Y with distribution $\mathrm{P}^{(-\infty)}$, $\sum_{j\in J} \delta_{(s_j, X^j)}$ is a Poisson point measure with intensity:

$$2\, \mathbf{1}_{\{s<0\}} \alpha(Y_s)\, ds\, \mathbf{1}_{\{H_{max}(X) < -s\}}\, \mathbb{N}_{Y_s}[dX].$$

Remark 1.3. Considering a superprocess with homogeneous branching mechanism, the Q-process may be easily defined from the well known Q-process for the total mass process (see for instance [7] in the case of a general branching mechanism). Thus the recurrence condition imposed on the spatial motion is not necessary for Williams decomposition, but it seems more natural in order to get the asymptotic distribution at the extinction time.

Remark 1.4. The genealogy of X defined in Proposition 3.12 allows us to interpret the following probability measure $\mathrm{P}_x^{(B,t)}$ as the law of the ancestral lineage of an individual sampled at random at height t (see the Bismut decomposition, Proposition 4.4):

$$\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\mathrm{e}^{-\int_0^t ds\, \beta(Y_s)}}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\, \beta(Y_s)}\right]}.$$

We prove in Lemma 6.13 that, if ϕ_0 is bounded from below and above by positive constants and that the operator $(\beta - \lambda_0) - \mathcal{L}$ is product critical, then the ancestral lineage of an individual sampled at random at height t under $\mathbb{N}_x$ converges as $t \to \infty$ to the law of the spine, that is $\mathrm{P}_x^{(B,t)}$ converges weakly to $\mathrm{P}_x^{\phi_0}$. This Feynman-Kac type penalization result (see Chapter 2 of Roynette and Yor [29]) heavily relies on the product criticality assumption, but holds without restriction on the sign of λ_0. It may be interpreted as an example of the so called globular state in random polymers, investigated in Cranston, Koralov and Molchanov [8].

Outline. We give some background on superprocesses with a non-homogeneous branching mechanism in the Section 2. Section 3 begins with the definition of the h-transform in the sense of Engländer and Pinsky, Definition 3.4, goes on with a Girsanov Theorem, Proposition 3.7, and ends up with the definition of the genealogy, Proposition 3.12, by combining both tools. Section 4 is mainly devoted to the proof of the William's decomposition, Theorem 4.12. By the way, we give a decomposition with respect to a randomly chosen individual, also known as a Bismut decomposition, in Proposition 4.2. Section 5 gives some applications of the Williams' decomposition: We first prove in Lemma 5.1 that the limit of the superprocesses conditioned to extinct *at* a remote time coincide with the Q-process (the superprocess conditioned to extinct *after* a remote time) and actually show in Theorem 5.5 that such a limit exists. We also consider in Theorem 5.9 the convergence of the process seen from the top (so, backward from the extinction time). All previous results are provided with a set of assumptions. We then give in Section 6 sufficient conditions for these assumptions to be valid in term of the generalized eigenvector and eigenvalue, then check they hold in Section 7 in two examples: the finite state space superprocess (with mass process the multitype Feller diffusion) and the superdiffusion.

2. Notations and definitions

This section, based on the lecture notes of Perkins [26], provides us with basic material about superprocesses, relying on their characterization via the Log Laplace equation.

We first introduce some definitions:

- (E, δ) is a Polish space, $\mathcal{B}$ its Borel sigma-field.
- $\mathcal{E}$ is the set of real valued measurable functions and $b\mathcal{E} \subset \mathcal{E}$ the subset of bounded functions.
- $\mathcal{C}(E, \mathbb{R})$, or simply $\mathcal{C}$, is the set of continuous real valued functions on E, $\mathcal{C}_b \subset \mathcal{C}$ the subset of continuous bounded functions.
- $D(\mathbb{R}^+, E)$, or simply D, is the set of càdlàg paths of E equipped with the Skorokhod topology, $\mathcal{D}$ is the Borel sigma field on D, and $\mathcal{D}_t$ the canonical right continuous filtration on D.
- For each set of functions, the superscript $.^+$ will denote the subset of the non-negative functions: For instance, $b\mathcal{E}^+$ stands for the subset of non negative functions of $b\mathcal{E}$.
- $\mathcal{M}_f(E)$ is the space of finite measures on E. The standard inner product notation will be used: for $g \in \mathcal{E}$ integrable with respect to $M \in \mathcal{M}_f(E)$, $M(g) = \int_E M(dx)g(x)$.

We can now introduce the two main ingredients which enter in the definition of a superprocess, the spatial motion and the branching mechanism:

- Assume $Y = (D, \mathcal{D}, \mathcal{D}_t, Y_t, \mathrm{P}_x)$ is a Borel strong Markov process. "Borel" means that $x \to \mathrm{P}_x(A)$ is $\mathcal{B}$ measurable for all $A \in \mathcal{B}$. Let E_x denote the expectation operator, and $(P_t, t \geq 0)$ the semi-group defined by: $P_t(f)(x) = \mathrm{E}_x[f(Y_t)]$. We impose the additional assumption that $P_t : \mathcal{C}_b \to \mathcal{C}_b$. In particular the process Y has no fixed discontinuities. The generator associated to the semi-group will be denoted $\mathcal{L}$. Remember f belongs to the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ if $f \in \mathcal{C}_b$ and for some $g \in \mathcal{C}_b$,

$$f(Y_t) - f(x) - \int_0^t ds\ g(Y_s) \text{ is a } P_x \text{ martingale for all } x \text{ in } E, \tag{1}$$

  in which case $g = \mathcal{L}(f)$.
- The functions α and β being elements of $\mathcal{C}_b$, with α bounded from below by a positive constant, the non-homogeneous quadratic branching mechanism $\psi^{\beta,\alpha}$ is defined by:

$$\psi^{\beta,\alpha}(x, \lambda) = \beta(x)\lambda + \alpha(x)\lambda^2, \tag{2}$$

  for all $x \in E$ and $\lambda \in \mathbb{R}$. We will just write ψ for $\psi^{\beta,\alpha}$ when there is no possible confusion. If α and β are constant functions, we will call the branching mechanism (and by extension, the corresponding superprocess) homogeneous.

The mild form of the Log Laplace equation is given by the integral equation, for $\phi, f \in b\mathcal{E}^+$, $t \geq 0$, $x \in E$:

$$u_t(x) + \mathrm{E}_x\left[\int_0^t ds\ \psi(Y_s, u_{t-s}(Y_s))\right] = \mathrm{E}_x\left[f(Y_t) + \int_0^t ds\ \phi(Y_s)\right]. \tag{3}$$

Theorem 2.1. *([26], Theorem II.5.11) Let $\phi, f \in b\mathcal{E}^+$. There is a unique jointly (in t and x) Borel measurable solution $u_t^{f,\phi}(x)$ of equation (3) such that $u_t^{f,\phi}$ is bounded on $[0,T]\times E$ for all $T>0$. Moreover, $u_t^{f,\phi} \geq 0$ for all $t \geq 0$.*

We shall write u^f for $u^{f,0}$ when ϕ is null.

We introduce the canonical space of continuous applications from $[0,\infty)$ to $\mathcal{M}_f(E)$, denoted by $\Omega := \mathcal{C}(\mathbb{R}^+, \mathcal{M}_f(E))$, endowed with its Borel sigma field $\mathcal{F}$, and the canonical right continuous filtration $\mathcal{F}_t$. Notice that $\mathcal{F} = \mathcal{F}_\infty$.

Theorem 2.2. *([26], Theorem II.5.11) Let $u_t^{f,\phi}(x)$ denote the unique jointly Borel measurable solution of equation (3) such that $u_t^{f,\phi}$ is bounded on $[0,T]\times E$ for all $T>0$. There exists a unique Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, (\mathbb{P}_\nu^{(\mathcal{L},\beta,\alpha)}, \nu \in \mathcal{M}_f(E)))$ such that:*

$$\forall \phi, f \in b\mathcal{E}^+, \qquad \mathbb{E}_\nu^{(\mathcal{L},\beta,\alpha)}\left[\mathrm{e}^{-X_t(f) - \int_0^t ds\ X_s(\phi)}\right] = \mathrm{e}^{-\nu(u_t^{f,\phi})}. \tag{4}$$

X is called the $(\mathcal{L},\beta,\alpha)$-superprocess.

We now state the existence theorem of the canonical measures:

Theorem 2.3. *([26], Theorem II.7.3) There exists a measurable family of σ-finite measures $(\mathbb{N}_x^{(\mathcal{L},\beta,\alpha)}, x \in E)$ on $(\Omega, \mathcal{F})$ which satisfies the following properties: If $\sum_{j\in\mathcal{J}} \delta_{(x^j, X^j)}$ is a Poisson point measure on $E\times\Omega$ with intensity $\nu(dx)\ \mathbb{N}_x^{(\mathcal{L},\beta,\alpha)}$, then $\sum_{j\in\mathcal{J}} X^j$ is an $(\mathcal{L},\beta,\alpha)$-superprocess started at ν.*

We will often abuse notation by denoting $\mathbb{P}_\nu$ (resp. $\mathbb{N}_x$) instead of $\mathbb{P}_\nu^{(\mathcal{L},\beta,\alpha)}$ (resp. $\mathbb{N}_x^{(\mathcal{L},\beta,\alpha)}$), and $\mathbb{P}_x$ instead of $\mathbb{P}_{\delta_x}$ when starting from δ_x the Dirac mass at point x.

Let X be a $(\mathcal{L},\beta,\alpha)$-superprocess. The exponential formula for Poisson point measures yields the following equality:

$$\forall f \in b\mathcal{E}^+,\ \mathbb{N}_{x_0}\left[1 - \mathrm{e}^{-X_t(f)}\right] = -\log \mathbb{E}_{x_0}\left[\mathrm{e}^{-X_t(f)}\right] = u_t^f(x_0), \tag{5}$$

where u_t^f is (uniquely) defined by equation (4).

Denote $H_{\max}$ the extinction time of X:

$$H_{\max} = \inf\{t>0;\ X_t = 0\}. \tag{6}$$

Definition 2.4 (Global extinction)**.** *The superprocess X suffers global extinction if $\mathbb{P}_\nu(H_{max} < \infty) = 1$ for all $\nu \in \mathcal{M}_f(E)$.*

We will need the the following assumption:

$(H1)$ **The $(\mathcal{L},\beta,\alpha)$-superprocess satisfies the global extinction property.**

We shall be interested in the function

$$v_t(x) = \mathbb{N}_x[H_{\max} > t]. \tag{7}$$

We set $v_\infty(x) = \lim_{t\to\infty} \downarrow v_t(x)$. The global extinction property is easily stated using v_∞.

Lemma 2.5. *The global extinction property holds if and only if $v_\infty = 0$.*

See also Lemma 4.9 for other properties of the function v.

Proof. The exponential formula for Poisson point measures yields:

$$\mathbb{P}_\nu(H_{\max} \leq t) = \mathrm{e}^{-\nu(v_t)}\,.$$

To conclude, let t goes to infinity in the previous equality to get:

$$\mathbb{P}_\nu(H_{\max} < \infty) = \mathrm{e}^{-\nu(v_\infty)}\,.$$

□

For homogeneous superprocesses (α and β constant), the function v is easy to compute and the global extinction holds if and only β is non-negative. Then, using stochastic domination argument, one get that a $(\mathcal{L}, \beta, \alpha)$-superprocess, with β non-negative, exhibits global extinction (see [16] p.80 for details).

3. A genealogy for the spatially dependent superprocess

We first recall (Section 3.1) the h-transform for superprocess introduced in [17] and then (Section 3.2) a Girsanov theorem previously introduced in [26] for interactive superprocesses. Those two transformations allow us to give a Radon-Nikodym derivative of the distribution of a superprocess with non-homogeneous branching mechanism with respect to the distribution of a superprocess with an homogeneous branching mechanism. The genealogy of the superprocess with an homogeneous branching mechanism can be described using a Brownian snake, see [12]. Then, in Section 3.3, we use the Radon-Nikodym derivative to transport this genealogy and get a genealogy for the superprocess with non-homogeneous branching mechanism.

3.1. ***h*-transform for superprocesses.** We first introduce a new probability measure on $(D, \mathcal{D})$ using the next Lemma.

Lemma 3.1. *Let g be a positive function of $\mathcal{D}(\mathcal{L})$ such that g is bounded from below by a positive constant. Then, the process $\big(\frac{g(Y_t)}{g(x)}\,\mathrm{e}^{-\int_0^t ds\ (\mathcal{L}g/g)(Y_s)}, t \geq 0\big)$ is a positive martingale under P_x.*

We set $\mathcal{D}_g(\mathcal{L}) = \{v \in \mathcal{C}_b,\ gv \in \mathcal{D}(\mathcal{L})\}$.

Proof. Let g be as in Lemma 3.1 and $f \in \mathcal{D}_g(\mathcal{L})$. The process:

$$\left((fg)(Y_t) - (fg)(x) - \int_0^t ds\ \mathcal{L}(fg)(Y_s),\ t \geq 0\right)$$

is a P_x martingale by definition of the generator $\mathcal{L}$. Thus, the process:

$$\left(\frac{(fg)(Y_t)}{g(x)} - f(x) - \int_0^t ds\ \frac{\mathcal{L}(fg)(Y_s)}{g(x)},\ t \geq 0\right)$$

is a P_x martingale. We set:

$$\begin{aligned}(8)\quad M_t^{f,g} = \mathrm{e}^{-\int_0^t ds\ (\mathcal{L}g/g)(Y_s)}\,&\frac{(fg)(Y_t)}{g(x)} - f(x)\\ &- \int_0^t ds\ \mathrm{e}^{-\int_0^s dr\ (\mathcal{L}g/g)(Y_r)}\left[\frac{\mathcal{L}(fg)(Y_s)}{g(x)} - \frac{\mathcal{L}(g)(Y_s)}{g(Y_s)}\frac{(fg)(Y_s)}{g(x)}\right].\end{aligned}$$

Itô's lemma then yields that the process $(M_t^{f,g}, t \geq 0)$ is another P_x martingale. Take f constant equal to 1 to get the result. □

Let P_x^g denote the probability measure on $(D, \mathcal{D})$ defined by:

$$\forall t \geq 0, \quad \frac{d\mathrm{P}^g_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{g(Y_t)}{g(x)}\, \mathrm{e}^{-\int_0^t ds\ (\mathcal{L}g/g)(Y_s)}\,. \tag{9}$$

Note that in the case where g is harmonic for the linear operator $\mathcal{L}$ (that is $\mathcal{L}g = 0$), the probability distribution P^g is the usual Doob h-transform of P for $h = g$.

We also introduce the generator $\mathcal{L}^g$ of the canonical process Y under P^g and the expectation operator E^g associated to P^g.

Lemma 3.2. *Let g be a positive function of $\mathcal{D}(\mathcal{L})$ such that g is bounded from below by a positive constant. Then, we have $\mathcal{D}_g(\mathcal{L}) \subset \mathcal{D}(\mathcal{L}^g)$ and*

$$\forall u \in \mathcal{D}_g(\mathcal{L}), \quad \mathcal{L}^g(u) = \frac{\mathcal{L}(gu) - \mathcal{L}(g)u}{g}\cdot$$

Proof. As, for $f \in \mathcal{D}_g(\mathcal{L})$, the process $(M_t^{f,g}, t \geq 0)$ defined by (8) is a martingale under P_x, we get that the process:

$$f(Y_t) - f(x) - \int_0^t ds\ \left(\frac{\mathcal{L}(fg)(Y_s) - \mathcal{L}(g)(Y_s)f(Y_s)}{g(Y_s)} \right), \quad t \geq 0$$

is a P_x^g martingale. This gives the result. □

Remark 3.3. Let $((t, x) \to g(t, x))$ be a function bounded from below by a positive constant, differentiable in t, such that $g(t, .) \in \mathcal{D}(\mathcal{L})$ for each t and $((t, x) \to \partial_t g(t, x))$ is bounded from above. By considering the process (t, Y_t) instead of Y_t, we have the immediate counterpart of Lemma 3.1 for time dependent function $g(t, .)$. In particular, we may define the following probability measure on $(D, \mathcal{D})$ (still denoted P_x^g by a small abuse of notations):

$$\forall t \geq 0, \quad \frac{d\mathrm{P}^g_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{g(t, Y_t)}{g(0, x)}\, \mathrm{e}^{-\int_0^t ds\ \frac{\mathcal{L}g + \partial_t g}{g}(s, Y_s)}, \tag{10}$$

where $\mathcal{L}$ acts on g as a function of x.

We now define the h-transform for superprocesses, as introduced in [17] (notice this does not correspond to the Doob h-transform for superprocesses).

Definition 3.4. *Let $X = (X_t, t \geq 0)$ be an $(\mathcal{L}, \beta, \alpha)$ superprocess. For $g \in b\mathcal{E}^+$, we define the h-transform of X (with $h = g$) as $X^g = (X_t^g, t \geq 0)$ the measure valued process given for all $t \geq 0$ by:*

$$X_t^g(dx) = g(x)X_t(dx). \tag{11}$$

Note that (11) holds point-wise, and that the law of the h-transform of a superprocess may be singular with respect to the law of the initial superprocess.

We first give an easy generalization of a result in section 2 of [17] for a general spatial motion.

Proposition 3.5. *Let g be a positive function of $\mathcal{D}(\mathcal{L})$ such that g is bounded from below by a positive constant. Then the process X^g is a $\left(\mathcal{L}^g, \frac{(-\mathcal{L}+\beta)g}{g}, \alpha g\right)$-superprocess.*

Proof. The Markov property of X^g is clear. We compute, for $f \in b\mathcal{E}^+$:

$$\mathbb{E}_x[\mathrm{e}^{-X_t^g(f)}] = \mathbb{E}_{\delta_x/g(x)}[\mathrm{e}^{-X_t(fg)}] = \mathrm{e}^{-u_t(x)/g(x)},$$

where, by Theorem 2.2, u satisfies:

$$u_t(x) + \mathrm{E}_x\Big[\int_0^t dr\ \psi(Y_r, u_{t-r}(Y_r))\Big] = \mathrm{E}_x\big[(fg)(Y_t)\big], \tag{12}$$

which can also be written:

$$u_t(x) + \mathrm{E}_x\Big[\int_0^s dr\ \psi(Y_r, u_{t-r}(Y_r))\Big] + \mathrm{E}_x\Big[\int_s^t dr\ \psi(Y_r, u_{t-r}(Y_r))\Big] = \mathrm{E}_x\big[(fg)(Y_t)\big].$$

But (12) written at time $t-s$ gives:

$$u_{t-s}(x) + \mathrm{E}_x\Big[\int_0^{t-s} dr\ \psi(Y_r, u_{t-s-r}(Y_r))\Big] = \mathrm{E}_x\big[(fg)(Y_{t-s})\big].$$

By comparing the two previous equations, we get:

$$u_t(x) + \mathrm{E}_x\Big[\int_0^s dr\ \psi(Y_r, u_{t-r}(Y_r))\Big] = \mathrm{E}_x\big[u_{t-s}(Y_s)\big],$$

and the Markov property now implies that the process:

$$u_{t-s}(Y_s) - \int_0^s dr\ \psi(Y_r, u_{t-r}(Y_r))$$

with $s \in [0,t]$ is a P_x martingale. Itô's lemma now yields that the process:

$$u_{t-s}(Y_s)\,\mathrm{e}^{-\int_0^s dr(\mathcal{L}g/g)(Y_r)} - \int_0^s dr\ \ \mathrm{e}^{-\int_0^r du\ (\mathcal{L}g/g)(Y_u)}\big(\psi(Y_r, u_{t-r}(Y_r)) - (\mathcal{L}g/g)(Y_r)\ u_{t-r}(Y_r)\big)$$

with $s \in [0,t]$ is another P_x martingale (the integrability comes from the assumption $\mathcal{L}g \in \mathcal{C}_b$ and $1/g \in \mathcal{C}_b$). Taking expectations at time $s=0$ and at time $s=t$, we have:

$$\begin{aligned} u_t(x) + \mathrm{E}_x\Big[\int_0^t ds\ \ \mathrm{e}^{-\int_0^s dr(\mathcal{L}g/g)(Y_r)}\big(\psi(Y_s, u_{t-s}(Y_s)) - (\mathcal{L}g/g)(Y_s)u_{t-s}(Y_s)\big)\Big] \\ = \mathrm{E}_x\Big[\mathrm{e}^{-\int_0^t dr(\mathcal{L}g/g)(Y_r)}(fg)(Y_t)\Big]. \end{aligned}$$

We divide both sides by $g(x)$ and expand ψ according to its definition:

$$\begin{aligned} \big(\frac{u_t}{g}\big)(x) + \mathrm{E}_x\Big[\int_0^t ds\ \frac{g(Y_s)}{g(x)}\,\mathrm{e}^{-\int_0^s dr(\mathcal{L}g/g)(Y_r)}\Big((\alpha g)(Y_s)\big(\frac{u_{t-s}}{g}\big)^2(Y_s) + \big(\beta - \frac{\mathcal{L}g}{g}\big)(Y_s)\big(\frac{u_{t-s}}{g}\big)(Y_s)\Big)\Big] \\ = \mathrm{E}_x\Big[\frac{g(Y_t)}{g(x)}\,\mathrm{e}^{-\int_0^t dr(\mathcal{L}g/g)(Y_r)}\ f(Y_t)\Big]. \end{aligned}$$

By definition of P_x^g from (9), we get that:

$$\big(\frac{u_t}{g}\big)(x) + \mathrm{E}_x^g\Big[\int_0^t ds\ \Big((\alpha g)(Y_s)\big(\frac{u_{t-s}}{g}\big)^2(Y_s) + \big(\beta - \frac{\mathcal{L}g}{g}\big)(Y_s)\big(\frac{u_{t-s}}{g}\big)(Y_s)\Big]\Big) = \mathrm{E}_x^g\big[f(Y_t)\big].$$

We conclude from Theorem 2.2 that X^g is a $(\mathcal{L}^g, \frac{(-\mathcal{L}+\beta)g}{g}, \alpha g)$-superprocess. □

In order to perform the h-transform of interest, we shall consider the following assumption.

$(H2)$ $1/\alpha$ **belongs to** $\mathcal{D}(\mathcal{L})$**.**

Notice that $(H2)$ implies that $\alpha\mathcal{L}(1/\alpha) \in \mathcal{C}_b$. Proposition 3.5 and Lemma 3.1 then yield the following Corollary.

Corollary 3.6. *Let X be an $(\mathcal{L},\beta,\alpha)$-superprocess. Assume $(H2)$. The process $X^{1/\alpha}$ is an $(\tilde{\mathcal{L}},\tilde{\beta},1)$-superprocess with:*

$$\tilde{\mathcal{L}} = \mathcal{L}^{1/\alpha} \quad \textit{and} \quad \tilde{\beta} = \beta - \alpha\mathcal{L}(1/\alpha). \tag{13}$$

Moreover, for all $t \geq 0$, the law $\tilde{\mathrm{P}}_x$ of the process Y with generator $\tilde{\mathcal{L}}$ is absolutely continuous on $\mathcal{D}_t$ with respect to P_x and its Radon-Nikodym derivative is given by:

$$\frac{d\tilde{\mathrm{P}}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\alpha(x)}{\alpha(Y_t)}\, \mathrm{e}^{\int_0^t ds\ (\tilde{\beta}-\beta)(Y_s)}\,. \tag{14}$$

We will note $\tilde{\mathbb{P}}$ for the law of $X^{1/\alpha}$ on the canonical space (that is $\tilde{\mathbb{P}} = \mathbb{P}^{(\tilde{\mathcal{L}},\tilde{\beta},1)}$) and $\tilde{\mathbb{N}}$ for its canonical measure. Observe that the branching mechanism of X under $\tilde{\mathbb{P}}$, which we shall write $\tilde{\psi}$, is given by:

$$\tilde{\psi}(x,\lambda) = \tilde{\beta}(x)\ \lambda + \lambda^2, \tag{15}$$

and the quadratic coefficient is no more dependent on x. Notice that $\mathbb{P}_{\alpha\nu}(X \in \cdot) = \tilde{\mathbb{P}}_\nu(\alpha X \in \cdot)$. This implies the following relationship on the canonical measures (use Theorem 2.3 to check it):

$$\alpha(x)\mathbb{N}_x[X \in \cdot] = \tilde{\mathbb{N}}_x[\alpha X \in \cdot]. \tag{16}$$

Recall that $v_t(x) = \mathbb{N}_x[H_{\max} > t] = \mathbb{N}_x[X_t \neq 0]$. We set $\tilde{v}_t(x) = \tilde{\mathbb{N}}_x[X_t \neq 0]$. As α is positive, equality (16) implies in particular that, for all $t > 0$ and $x \in E$:

$$\alpha(x)v_t(x) = \tilde{v}_t(x). \tag{17}$$

3.2. **A Girsanov type theorem.**

The following assumption will be used to perform the Girsanov change of measure.

$(H3)$ **Assume $(H2)$ holds. The function $\tilde{\beta}$ defined in (13) is in $\mathcal{D}(\tilde{\mathcal{L}})$, with $\tilde{\mathcal{L}}$ defined in (13).**

For $z \in \mathbb{R}$, we set $z_+ = \max(z,0)$. Under $(H2)$ and $(H3)$, we define:

$$\beta_0 = \sup_{x\in E} \max\left(\tilde{\beta}(x), \sqrt{(\tilde{\beta}^2(x) - 2\tilde{\mathcal{L}}(\tilde{\beta})(x))_+}\right) \quad \text{and} \quad q(x) = \frac{\beta_0 - \tilde{\beta}(x)}{2}. \tag{18}$$

Notice that $q \geq 0$.

We shall consider the distribution of the homogeneous $(\tilde{\mathcal{L}},\beta_0,1)$-superprocess, which we will denote by $\mathbb{P}^0$ ($\mathbb{P}^0 = \mathbb{P}^{(\tilde{\mathcal{L}},\beta_0,1)}$) and its canonical measure $\mathbb{N}^0$. Note that the branching mechanism of X under $\mathbb{P}^0$ is homogeneous (the branching mechanism does not depend on x). We set ψ^0 for $\psi^{\beta_0,1}$. Since ψ^0 does not depend anymore on x we shall also write $\psi^0(\lambda)$ for $\psi^0(x,\lambda)$:

$$\psi^0(\lambda) = \beta_0\lambda + \lambda^2. \tag{19}$$

Proposition 3.7 below is a Girsanov's type theorem which allows us to finally reduce the distribution $\tilde{\mathbb{P}}$ to the homogeneous distribution $\mathbb{P}^0$. We introduce the process $M = (M_t, t \geq 0)$ defined by:

$$M_t = \exp\left(X_0(q) - X_t(q) - \int_0^t ds\ X_s(\varphi)\right), \tag{20}$$

where the function φ is defined by:

$$\varphi(x) = \tilde{\psi}(x,q(x)) - \tilde{\mathcal{L}}(q)(x), \quad x \in E. \tag{21}$$

Proposition 3.7. *A Girsanov's type theorem. Assume* $(H2)$ *and* $(H3)$ *hold. Let* X *be a* $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$*-superprocess.*

(i) *The process* M *is a bounded* $\mathcal{F}$*-martingale under* $\tilde{\mathbb{P}}_\nu$ *which converges a.s. to*

$$M_\infty = \mathrm{e}^{X_0(q) - \int_0^{+\infty} ds\ X_s(\varphi)}\ \mathbf{1}_{\{H_{max} < +\infty\}}.$$

(ii) *We have:*

$$\frac{d\mathbb{P}^0_\nu}{d\tilde{\mathbb{P}}_\nu} = M_\infty.$$

(iii) *If moreover* $(H1)$ *holds, then* $\mathbb{P}^0_\nu$*-a.s. we have* $M_\infty > 0$*, the probability measure* $\tilde{\mathbb{P}}_\nu$ *is absolutely continuous with respect to* $\mathbb{P}^0_\nu$ *on* $\mathcal{F}$*:*

$$\frac{d\tilde{\mathbb{P}}_\nu}{d\mathbb{P}^0_\nu} = \frac{1}{M_\infty}, \quad \textit{and} \quad \frac{d\tilde{\mathbb{N}}_x}{d\mathbb{N}^0_x} = \mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)}.$$

We also have:

$$q(x) = \mathbb{N}^0_x\left[\mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)} - 1\right]. \tag{22}$$

The two first points are a particular case of Theorem IV.1.6 p.252 in [26] on interactive drift. For the sake of completeness, we give a proof based on the mild form of the Log Laplace equation (3) introduced in Section 2. Notice that:

$$\psi^0(\lambda) = \tilde{\psi}(x, \lambda + q(x)) - \tilde{\psi}(x, q(x)). \tag{23}$$

Thus, Proposition 3.7 appears as a non-homogeneous generalization of Corollary 4.4 in [2]. We first give an elementary Lemma.

Lemma 3.8. *Assume* $(H2)$ *and* $(H3)$ *hold. The function* φ *defined by (21) is non-negative.*

Proof. The following computation:

$$\begin{aligned}\varphi(x) = \tilde{\psi}(x, q(x)) - \tilde{\mathcal{L}}(q)(x) &= q(x)^2 + \tilde{\beta} q(x) - \tilde{\mathcal{L}}(q)(x)\\ &= \left(\frac{\beta_0 - \tilde{\beta}(x)}{2}\right)^2 + \tilde{\beta}(x)\ \frac{\beta_0 - \tilde{\beta}(x)}{2} - \tilde{\mathcal{L}}(q)(x)\\ &= \frac{\beta_0^2 - \tilde{\beta}^2(x) + 2\tilde{\mathcal{L}}(\tilde{\beta})(x)}{4}\end{aligned}$$

and the definition (18) of β_0 ensure that the function φ is non-negative. □

Proof of Proposition 3.7. First observe that M is $\mathcal{F}$-adapted. As the function q also is non-negative, we deduce from Lemma 3.8 that the process M is bounded by $\mathrm{e}^{X_0(q)}$.

Let $f \in b\mathcal{E}^+$. On the one hand, we have:

$$\tilde{\mathbb{E}}_x[M_t\,\mathrm{e}^{-X_t(f)}] = \tilde{\mathbb{E}}_x[\mathrm{e}^{q(x) - X_t(q+f) - \int_0^t ds\ X_s(\varphi)}] = \mathrm{e}^{q(x) - r_t(x)},$$

where, according to Theorem 2.2, $r_t(x)$ is bounded on $[0,T] \times E$ for all $T > 0$ and satisfies:

$$\begin{aligned} r_t(x) + \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s}, r_s(Y_{t-s}))\right]& \\ = \tilde{\mathrm{E}}_x\Bigg[\int_0^t ds\ \big(\tilde{\psi}(Y_{t-s}, q(Y_{t-s})) &- \tilde{\mathcal{L}}(q)(Y_{t-s})\big) + (q+f)(Y_t)\Bigg]. \end{aligned} \tag{24}$$

On the other hand, we have
$$\mathbb{E}^0_x[\mathrm{e}^{-X_t(f)}] = \mathrm{e}^{-w_t(x)},$$
where $w_t(x)$ is bounded on $[0,T]\times E$ for all $T>0$ and satisfies:
$$w_t(x) + \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \psi^0(Y_{t-s}, w_s(Y_{t-s}))\right] = \tilde{\mathrm{E}}_x[f(Y_t)].$$
Using (23), rewrite the previous equation under the form:
$$(25)\qquad w_t(x) + \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s}, (w_s+q)(Y_{t-s}))\right] = \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s}, q(Y_{t-s})) + f(Y_t)\right].$$
We now make use of the Dynkin's formula with $(H3)$:
$$(26)\qquad q(x) = -\tilde{\mathrm{E}}_x\left[\int_0^t \tilde{\mathcal{L}}(q)(Y_s)\right] + \tilde{\mathrm{E}}_x[q(Y_t)],$$
and sum the equations (25) and (26) term by term to get:
$$(27)\quad (w_t+q)(x) + \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s}, (w_s+q)(Y_{t-s}))\right] = \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \left(\tilde{\psi}(Y_{t-s}, q(Y_{t-s})) - \tilde{\mathcal{L}}(q)(Y_{t-s})\right) + (q+f)(Y_t)\right].$$

The functions $r_t(x)$ and $w_t(x)+q(x)$ are bounded on $[0,T]\times E$ for all $T>0$ and satisfy the same equation, see equations (24) and (27). By uniqueness, see Theorem 2.1, we finally get that $w_t + q = r_t$. This gives:
$$(28)\qquad \tilde{\mathbb{E}}_x[M_t\,\mathrm{e}^{-X_t(f)}] = \mathbb{E}^0_x[\mathrm{e}^{-X_t(f)}].$$
The Poissonian decomposition of the superprocesses, see Theorem 2.3, and the exponential formula enable us to extend this relation to arbitrary initial measures ν:
$$(29)\qquad \tilde{\mathbb{E}}_\nu[M_t\,\mathrm{e}^{-X_t(f)}] = \mathbb{E}^0_\nu[\mathrm{e}^{-X_t(f)}].$$
This equality with $f=1$ and the Markov property of X proves the first part of item (i).

Now, a direct induction based on the Markov property yields that, for all positive integer n, and $f_1,\ldots,f_n \in b\mathcal{E}^+$, $0\le s_1\le\ldots\le s_n\le t$:
$$(30)\qquad \tilde{\mathbb{E}}_\nu[M_t\,\mathrm{e}^{-\sum_{1\le i\le n} X_{s_i}(f_i)}] = \mathbb{E}^0_\nu[\mathrm{e}^{-\sum_{1\le i\le n} X_{s_i}(f_i)}].$$
And we conclude with an application of the monotone class theorem that, for all non-negative $\mathcal{F}_t$-measurable random variable Z:
$$\tilde{\mathbb{E}}_\nu[M_t Z] = \mathbb{E}^0_\nu[Z].$$
The martingale M is bounded and thus converges a.s. to a limit M_∞. We deduce that for all non-negative $\mathcal{F}_t$-measurable random variable Z:
$$(31)\qquad \tilde{\mathbb{E}}_\nu[M_\infty Z] = \mathbb{E}^0_\nu[Z].$$
This also holds for any non-negative $\mathcal{F}_\infty$-measurable random variable Z. This gives the second item (ii).

On $\{H_{\max} < +\infty\}$, then clearly M_t converges to $\mathrm{e}^{X_0(q) - \int_0^{+\infty} ds\, X_s(\varphi)}$. Notice that $\mathbb{P}^0_\nu(H_{\max} = +\infty) = 0$. We deduce from (31) with $Z = \mathbf{1}_{\{H_{\max}=+\infty\}}$ that $\tilde{\mathbb{P}}_\nu$-a.s. on $\{H_{\max}=+\infty\}$, $M_\infty = 0$. This gives the last part of item (i).

Now, we prove the third item (iii). Notice that (31) implies that $\mathbb{P}^0_\nu$-a.s. $M_\infty > 0$. Thanks to $(H1)$, we also have that $\tilde{\mathbb{P}}_\nu$-a.s. $M_\infty > 0$. Let Z be a non-negative $\mathcal{F}_\infty$-measurable random variable. Applying (31) with Z replaced by $\mathbf{1}_{\{M_\infty>0\}}Z/M_\infty$, we get:

$$\tilde{\mathbb{E}}_\nu[Z] = \tilde{\mathbb{E}}_\nu\left[M_\infty \mathbf{1}_{\{M_\infty>0\}}\frac{Z}{M_\infty}\right] = \mathbb{E}^0_\nu\left[\frac{Z}{M_\infty}\mathbf{1}_{\{M_\infty>0\}}\right] = \mathbb{E}^0_\nu\left[\frac{Z}{M_\infty}\right].$$

This gives the first part of item (iii).

Notice that for all positive integer n, and $f_1, \ldots, f_n \in b\mathcal{E}^+$, $0 \le s_1 \le \ldots \le s_n$, we have

$$\begin{aligned}
\tilde{\mathbb{N}}_x\left[1 - \mathrm{e}^{\sum_{1\le i\le n} X_{s_i}(f_i)}\right] &= -\log\left(\tilde{\mathrm{E}}_x\left[\mathrm{e}^{\sum_{1\le i\le n} X_{s_i}(f_i)}\right]\right)\\
&= -\log\left(\mathrm{E}^0_x\left[\mathrm{e}^{\sum_{1\le i\le n} X_{s_i}(f_i)+\int_0^{+\infty} ds\, X_s(\varphi)}\right]\right) + q(x)\\
&= \mathbb{N}^0_x\left[1 - \mathrm{e}^{\sum_{1\le i\le n} X_{s_i}(f_i)+\int_0^{+\infty} ds\, X_s(\varphi)}\right] + q(x).
\end{aligned}$$

Taking $f_i = 0$ for all i gives (22). This implies:

$$\tilde{\mathbb{N}}_x\left[1 - \mathrm{e}^{\sum_{1\le i\le n} X_{s_i}(f_i)}\right] = \mathbb{N}^0_x\left[\mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)}\left(1 - \mathrm{e}^{\sum_{1\le i\le n} X_{s_i}(f_i)}\right)\right].$$

The monotone class theorem gives then the last part of item (iii). □

3.3. **Genealogy for superprocesses.** We now recall the genealogy of X under $\mathbb{P}^0$ given by the Brownian snake from [12]. We assume $(H2)$ and $(H3)$ hold.

Let $\mathcal{W}$ denote the set of all càdlàg killed paths in E. An element $w \in \mathcal{W}$ is a càdlàg path: $w : [0, \eta(w)) \to E$, with $\eta(w)$ the lifetime of the path w. By convention the trivial path $\{x\}$, with $x \in E$, is a killed path with lifetime 0 and it belongs to $\mathcal{W}$. The space $\mathcal{W}$ is Polish for the distance:

$$d(w, w') = \delta(w(0), w(0)') + |\eta(w) - \eta(w')| + \int_0^{\eta(w)\wedge\eta(w')} ds\; d_s(w_{[0,s]}, w'_{[0,s]}),$$

where d_s refers to the Skorokhod metric on the space $D([0, s], E)$, and w_I is the restriction of w on the interval I. Denote $\mathcal{W}_x$ the set of stopped paths w such that $w(0) = x$. We work on the canonical space of continuous applications from $[0, \infty)$ to $\mathcal{W}$, denoted by $\bar{\Omega} := \mathcal{C}(\mathbb{R}^+, \mathcal{W})$, endowed with the Borel sigma field $\bar{\mathcal{G}}$ for the distance d, and the canonical right continuous filtration $\bar{\mathcal{G}}_t = \sigma\{W_s, s \le t\}$, where $(W_s, s \in \mathbb{R}^+)$ is the canonical coordinate process. Notice $\bar{\mathcal{G}} = \bar{\mathcal{G}}_\infty$ by construction. We set $H_s = \eta(W_s)$ the lifetime of W_s.

Definition 3.9 (Proposition 4.1.1 and Theorem 4.1.2 of [12])**.** *Fix $W_0 \in \mathcal{W}_x$. There exists a unique $\mathcal{W}_x$-valued Markov process $W = (\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathcal{G}}_t, W_t, \mathbf{P}^0_{W_0})$, called the Brownian snake, starting at W_0 and satisfying the two properties:*

- (i) *The lifetime process $H = (H_s, s \ge 0)$ is a reflecting Brownian motion with non-positive drift $-\beta_0$, starting from $H_0 = \eta(W_0)$.*
- (ii) *Conditionally given the lifetime process H, the process $(W_s, s \ge 0)$ is distributed as an inhomogeneous Markov process, with transition kernel specified by the two following prescriptions, for $0 \le s \le s'$:*
  - *$W_{s'}(t) = W_s(t)$ for all $t < H_{[s,s']}$, with $H_{[s,s']} = \inf_{s\le r\le s'} H_r$.*
  - *Conditionally on $W_s(H_{[s,s']}-)$, the path $\left(W_{s'}(H_{[s,s']} + t), 0 \le t < H_{s'} - H_{[s,s']}\right)$ is independent of W_s and is distributed as $Y_{[0, H_{s'} - H_{[s,s']})}$ under $\tilde{\mathrm{P}}_{W_s(H_{[s,s']}-)}$.*

This process will be called the β_0-snake started at W_0, and its law denoted by $\mathbf{P}^0_{W_0}$.

We will just write $\mathbf{P}^0_x$ for the law of the snake started at the trivial path $\{x\}$. The corresponding excursion measure $\mathbf{N}^0_x$ of W is given as follows: the lifetime process H is distributed according to the Itô measure of the positive excursion of a reflecting Brownian motion with non-positive drift $-\beta_0$, and conditionally given the lifetime process H, the process $(W_s, s \geq 0)$ is distributed according to (ii) of Definition 3.9. Let

$$\sigma = \inf\{s > 0; H_s = 0\}$$

denote the length of the excursion under $\mathbf{N}^0_x$.

Let $(l^r_s, r \geq 0, s \geq 0)$ be the bicontinuous version of the local time process of H; where l^r_s refers to the local time at level r at time s. We also set $\hat{w} = w(\eta(w)-)$ for the left end position of the path w. We consider the measure valued process $X(W) = (X_t(W), t \geq 0)$ defined under $\mathbf{N}^0_x$ by:

$$X_t(W)(dx) = \int_0^\sigma d_s l^t_s \, \delta_{\hat{W}_s}(dx). \tag{32}$$

The β_0-snake gives the genealogy of the $(\tilde{\mathcal{L}}, \beta_0, 1)$ superprocess in the following sense.

Proposition 3.10 ([12], Theorem 4.2.1)**.** *We have:*

- *The process $X(W)$ is under $\mathbf{N}^0_x$ distributed as X under $\mathbb{N}^0_x$.*
- *Let $\sum_{j\in\mathcal{J}} \delta_{(x^j, W^j)}$ be a Poisson point measure on $E \times \bar{\Omega}$ with intensity $\nu(dx)\, \mathbf{N}^0_x[dW]$. Then $\sum_{j\in\mathcal{J}} X(W^j)$ is an $(\tilde{\mathcal{L}}, \beta_0, 1)$-superprocess started at ν.*

Notice that, under $\mathbf{N}^0_x$, the extinction time of $X(W)$ is defined by

$$\inf\{t; X_t(W) = 0\} = \sup_{s\in[0,\sigma]} H_s,$$

and we shall write this quantity $H_{\max}$ or $H_{\max}(W)$ if we need to stress the dependence in W. This notation is coherent with (6).

We now transport the genealogy of X under $\mathbb{N}^0$ to a genealogy of X under $\tilde{\mathbb{N}}$. In order to simplify notations, we shall write X for $X(W)$ when there is no confusion.

Definition 3.11. *Under (H1)-(H3), we define a measure $\tilde{\mathbf{N}}_x$ on $(\bar{\Omega}, \bar{\mathcal{G}})$ by:*

$$\forall W \in \bar{\Omega}, \quad \frac{d\tilde{\mathbf{N}}_x}{d\mathbf{N}^0_x}(W) = \frac{d\tilde{\mathbb{N}}_x}{d\mathbb{N}^0_x}(X(W)) = \frac{1}{M_\infty} = \mathrm{e}^{\int_0^{+\infty} ds\; X_s(\varphi)}.$$

Notice the second equality in the previous definition is the third item of Proposition 3.7.

At this point, the genealogy defined for X under $\tilde{\mathbb{N}}_x$ will give the genealogy of X under $\mathbb{N}$ up to a weight. We set

$$\mathbf{N}_x = \frac{1}{\alpha(x)} \tilde{\mathbf{N}}_x. \tag{33}$$

Proposition 3.12. *We have:*

(i) *X under $\tilde{\mathbf{N}}_x$ is distributed as X under $\tilde{\mathbb{N}}_x$.*

(ii) *The weighted process $X^{weight} = (X^{weight}_t, t \geq 0)$ with*

$$X^{weight}_t(dx) = \int_0^\sigma d_s l^t_s \; \alpha(\hat{W}_s)\delta_{\hat{W}_s}(dx), \quad t \geq 0, \tag{34}$$

is under $\mathbf{N}_x$ distributed as X under $\mathbb{N}_x$.

We may write $X^{\mathrm{weight}}(W)$ for X^{weight} to emphasize the dependence in the snake W.

Proof. This is a direct consequence of Definition 3.11 and (16). □

We shall say that W under $\mathbf{N}_x$ provides through (34) a genealogy for X under $\mathbb{N}_x$.

4. A Williams' decomposition

In Section 4.1, we give a decomposition of the genealogy of the superprocesses $(\mathcal{L}, \beta, \alpha)$ and $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$ with respect to a randomly chosen individual. In Section 4.2, we give a Williams' decomposition of the genealogy of the superprocesses $(\mathcal{L}, \beta, \alpha)$ and $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$ with respect to the last individual alive.

4.1. **Bismut's decomposition.** A decomposition of the genealogy of the homogeneous superprocess with respect to a randomly chosen individual is well known in the homogeneous case, even for a general branching mechanism (see lemmas 4.2.5 and 4.6.1 in [12]).

We now explain how to decompose the snake process under the excursion measure ($\tilde{\mathbf{N}}_x$ or $\mathbf{N}^0_x$) with respect to its value at a given time. Recall $\sigma = \inf\{s > 0, H_s = 0\}$ denote the length of the excursion. Fix a real number $t \in [0, \sigma]$. We consider the process $H^{(g)}$ (on the left of t) defined on $[0, t]$ by $H^{(g)}_s = H_{t-s} - H_t$ for all $s \in [0, t]$. The excursion intervals above 0 of the process $(H^{(g)}_s - \inf_{0\leq s'\leq s} H^{(g)}_{s'}, 0 \leq s \leq t)$ are denoted $\bigcup_{j\in J^{(g)}}(c_j, d_j)$. We also consider the process $H^{(d)}$ (on the right of t) defined on $[0, \sigma - t]$ by $H^{(d)}_s = H_{t+s} - H_t$. The excursion intervals above 0 of the process $(H^{(d)}_s - \inf_{0\leq s'\leq s} H^{(d)}_{s'}, 0 \leq s \leq \sigma - t)$ are denoted $\bigcup_{j\in J^{(d)}}(c_j, d_j)$. We define the level of the excursion j as $s_j = H_{t-c_j}$ if $j \in J^{(g)}$ and $s_j = H_{t+c_j}$ if $j \in J^{(d)}$. We also define for the excursion j the corresponding excursion of the snake: $W^j = (W^j_s, s \geq 0)$ as

$$W^j_s(.) = W_{t-(c_j+s)\wedge d_j}(. + s_j) \quad \text{if } j \in J^{(g)}, \text{ and } \quad W^j_s(.) = W_{t+(c_j+s)\wedge d_j}(. + s_j) \quad \text{if } j \in J^{(d)}.$$

We consider the following two point measures on $\mathbb{R}^+ \times \bar{\Omega}$: for $\varepsilon \in \{g, d\}$,

$$R^\varepsilon_t = \sum_{j\in J^{(\varepsilon)}} \delta_{(s_j, W^j)}. \tag{35}$$

Notice that under $\mathbf{N}^0_x$ (and under $\tilde{\mathbf{N}}_x$ if $(H1)$ holds), the process W can be reconstructed from the triplet (W_t, R^g_t, R^d_t). We are interested in the probabilistic structure of this triplet, when t is chosen according to the Lebesgue measure on the excursion time interval of the snake. Under $\mathbf{N}^0$, this result is as a consequence of Lemmas 4.2.4 and 4.2.5 from [12]. We recall this result in the next Proposition.

For a point measure $R = \sum_{j\in J} \delta_{(s_j, x_j)}$ on a space $\mathbb{R} \times \mathcal{X}$ and $A \subset \mathbb{R}$, we shall consider the restriction of R to $A \times \mathcal{X}$ given by $R_A = \sum_{j\in J} \mathbf{1}_A(s_j)\delta_{(s_j, x_j)}$.

Proposition 4.1 ([12], Lemmas 4.2.4 and 4.2.5)**.** *For every measurable non-negative function F, the following formulas hold:*

$$\mathbf{N}^0_x\Big[\int_0^\sigma ds\ F(W_s, R^g_s, R^d_s)\Big] = \int_0^\infty \mathrm{e}^{-\beta_0 r}\, dr\ \tilde{\mathrm{E}}_x\Big[F(Y_{[0,r)}, \hat{R}^{B,g}_{[0,r)}, \hat{R}^{B,d}_{[0,r)})\Big], \tag{36}$$

$$\mathbf{N}^0_x\Big[\int_0^\sigma d_s l^t_s\ F(W_s, R^g_s, R^d_s)\Big] = \mathrm{e}^{-\beta_0 t}\ \tilde{\mathrm{E}}_x\Big[F(Y_{[0,t)}, \hat{R}^{B,g}_{[0,t)}, \hat{R}^{B,d}_{[0,t)})\Big], \quad t > 0, \tag{37}$$

where under $\tilde{\mathrm{E}}_x$ and conditionally on Y, $\hat{R}^{B,g}$ and $\hat{R}^{B,d}$ are two independent Poisson point measures with intensity $\hat{\nu}^B(ds, dW) = ds\ \mathbf{N}^0_{Y_s}[dW]$.

The next Proposition gives a similar result in the non-homogeneous case.

Proposition 4.2. *Under* $(H1)$*-*$(H3)$*, for every measurable non-negative function* F*, the two formulas hold:*

$$\tilde{\mathbf{N}}_x\left[\int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\right] = \int_0^\infty dr\ \tilde{\mathrm{E}}_x\left[\mathrm{e}^{-\int_0^r ds\ \tilde{\beta}(Y_s)}\, F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\right], \tag{38}$$

where under $\tilde{\mathrm{E}}_x$ *and conditionally on* Y*,* $R^{B,g}_{[0,r)}$ *and* $R^{B,d}_{[0,r)}$ *are two independent Poisson point measures with intensity*

$$\nu^B(ds, dW) = ds\ \tilde{\mathbf{N}}_{Y_s}[dW] = ds\ \alpha(Y_s)\mathbf{N}_{Y_s}[dW]; \tag{39}$$

and

$$\mathbf{N}_x\left[\int_0^\sigma ds\ \alpha(\hat{W}_s)F(W_s, R_s^g, R_s^d)\right] = \int_0^\infty dr\ \mathrm{E}_x\left[\mathrm{e}^{-\int_0^r ds\ \beta(Y_s)}\, F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\right], \tag{40}$$

where under E_x *and conditionally on* Y*,* $R^{B,g}_{[0,r)}$ *and* $R^{B,d}_{[0,r)}$ *are two independent Poisson point measures with intensity* ν^B*.*

Observe there is a weight $\alpha(\hat{W}_s)$ in (40) (see also (34) where this weight appears) which modifies the law of the individual picked at random, changing the modified diffusion $\tilde{\mathrm{P}}_x$ in (38) into the original one P_x.

We shall use the following elementary Lemma on Poisson point measure.

Lemma 4.3. *Let* R *be a Poisson point measure on a Polish space with intensity* ν*. Let* f *be a non-negative measurable function* f *such that* $\nu(\mathrm{e}^f - 1) < +\infty$*. Then for any non-negative measurable function* F*, we have:*

$$\mathbb{E}\left[F(R)\,\mathrm{e}^{R(f)}\right] = \mathbb{E}\left[F(\tilde{R})\right]\mathrm{e}^{\nu(\mathrm{e}^f - 1)}, \tag{41}$$

where $\tilde{R}$ *is a Poisson point measure with intensity* $\tilde{\nu}(dx) = \mathrm{e}^{f(x)}\,\nu(dx)$*.*

Proof of Proposition 4.2. We keep notations introduced in Propositions 4.1 and 4.2. We have:

$$\begin{aligned}
\tilde{\mathbf{N}}_x&\left[\int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\right]\\
&= \mathbf{N}_x^0\left[\mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)}\int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\right]\\
&= \mathbf{N}_x^0\left[\int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\,\mathrm{e}^{(R_s^g + R_s^d)(f)}\right]\\
&= \int_0^\infty \mathrm{e}^{-\beta_0 r}\, dr\ \tilde{\mathrm{E}}_x\left[F(Y_{[0,r)}, \hat{R}^{B,g}_{[0,r)}, \hat{R}^{B,d}_{[0,r)})\,\mathrm{e}^{(\hat{R}^{B,g}_{[0,r)} + \hat{R}^{B,d}_{[0,r)})(f)}\right]\\
&= \int_0^\infty \mathrm{e}^{-\beta_0 r}\, dr\ \tilde{\mathrm{E}}_x\left[F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\,\mathrm{e}^{2\int_0^r ds\ \mathbf{N}^0_{Y_s}[\mathrm{e}^{\int_0^{+\infty} X_r(W)(\varphi)} - 1]}\right]\\
&= \int_0^\infty \mathrm{e}^{-\beta_0 r}\, dr\ \tilde{\mathrm{E}}_x\left[F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\,\mathrm{e}^{2\int_0^r ds\ q(Y_s)}\right]\\
&= \int_0^\infty dr\ \tilde{\mathrm{E}}_x\left[F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\,\mathrm{e}^{-\int_0^r ds\ \tilde{\beta}(Y_s)}\right],
\end{aligned}$$

where the first equality comes from $(H1)$ and item (iii) of Proposition 3.7, we set $f(s, W) = \int_0^{+\infty} X_r(W)(\varphi)$ for the second equality, we use Proposition 4.1 for the third equality, we use

Lemma 4.3 for the fourth, we use (22) for the fifth, and the definition (18) of q in the last. This proves (38).

Then replace $F(W_s, R_s^g, R_s^d)$ by $\alpha(\hat{W}_s)F(W_s, R_s^g, R_s^d)$ in (38) and use (14) as well as (33) to get (40). □

The proof of the following Proposition is similar to the proof of Proposition 4.2 and is not reproduced here.

Proposition 4.4. *Under $(H1)$-$(H3)$, for every measurable non-negative function F, the two formulas hold: for fixed $t > 0$,*

$$\tilde{\mathbf{N}}_x\left[\int_0^\sigma d_s l_s^t \; F(W_s, R_s^g, R_s^d)\right] = \tilde{\mathrm{E}}_x\left[\mathrm{e}^{-\int_0^t ds \; \tilde{\beta}(Y_s)} \, F(Y_{[0,t)}, R^{B,g}_{[0,t)}, R^{B,d}_{[0,t)})\right], \tag{42}$$

where under $\tilde{\mathrm{E}}_x$ and conditionally on Y, $R^{B,g}$ and $R^{B,d}$ are two independent Poisson point measures with intensity ν^B defined in (39), and

$$\mathbf{N}_x\left[\int_0^\sigma d_s l_s^t \; \alpha(\hat{W}_s)F(W_s, R_s^g, R_s^d)\right] = \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds \; \beta(Y_s)} \, F(Y_{[0,t)}, R^{B,g}_{[0,t)}, R^{B,d}_{[0,t)})\right], \tag{43}$$

where under E_x and conditionally on Y, $R^{B,g}$ and $R^{B,d}$ are two independent Poisson point measures with intensity ν^B.

As an example of application of this Proposition, we can recover easily the following well known result.

Corollary 4.5. *Under $(H1)$-$(H3)$, for every measurable non-negative functions f and g on E, we have:*

$$\mathbb{N}_x\left[X_t(f)\,\mathrm{e}^{-X_t(g)}\right] = \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds \; \partial_\lambda \psi\left(Y_s, \; \mathbb{N}_{Y_s}\left[1-\mathrm{e}^{X_{t-s}(g)}\right]\right)} \, f(Y_t)\right].$$

In particular, we recover the so-called "many-to-one" formula (with $g = 0$ in Corollary 4.5):

$$\mathbb{N}_x[X_t(f)] = \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds \; \beta(Y_s)} \, f(Y_t)\right]. \tag{44}$$

Remark 4.6. Equation (44) justifies the introduction of the following family of probability measures indexed by $t \geq 0$:

$$\frac{d\mathrm{P}^{(B,t)}_{x \;|\mathcal{D}_t}}{d\mathrm{P}_{x \;|\mathcal{D}_t}} = \frac{\mathrm{e}^{-\int_0^t ds \; \beta(Y_s)}}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds \; \beta(Y_s)}\right]}, \tag{45}$$

which can be understood as the law of the ancestral lineage of an individual sampled at random at height t under the excursion measure $\mathbb{N}_x$, and also correspond to Feynman Kac penalization of the original spatial motion P_x (see [29]). Notice that this law does not depend on the parameter α. These probability measures are not compatible as t varies but will be shown in Lemma 6.13 to converge as $t \to \infty$ in restriction to $\mathcal{D}_s$, s fixed, $s \leq t$, under some ergodic assumption (see $(H9)$ in Section 6).

Proof. We set for $w \in \mathcal{W}$ with $\eta(w) = t$ and r_1, r_2 two point measures on $\mathbb{R}^+ \times \bar{\Omega}$

$$F(w, r_1, r_2) = f(\hat{w})\,\mathrm{e}^{h(r_1)+h(r_2)},$$

where $h(\sum_{i\in I}\delta_{(s_i,W^i)}) = \sum_{s_i<t} X^{\text{weight}}(W^i)_{t-s_i}(g)$. We have:

$$\begin{aligned}
\mathbb{N}_x\left[X_t(f)\,\mathrm{e}^{-X_t(g)}\right] &= \mathbf{N}_x\left[\int_0^\sigma d_s l_s^t\ \alpha(\hat{W}_s)F(W_s,R_s^g,R_s^d)\right]\\
&= \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\,f(Y_t)\,\mathrm{e}^{h(R^{B,g}_{[0,r)})+h(R^{B,d}_{[0,r)})}\right]\\
&= \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\,f(Y_t)\,\mathrm{e}^{-\int_0^t 2\alpha(Y_s)\mathbf{N}_{Y_s}[1-\mathrm{e}^{X^{\text{weight}}_{t-s}(g)}]}\right]\\
&= \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \partial_\lambda\psi\left(Y_s,\ \mathbb{N}_{Y_s}\left[1-\mathrm{e}^{X_{t-s}(g)}\right]\right)}\,f(Y_t)\right],
\end{aligned}$$

where we used item (ii) of Proposition 3.12 for the first and last equality, (43) with F previously defined for the second, formula for exponentials of Poisson point measure and (33) for the third. □

4.2. **Williams' decomposition.** We first recall the Williams' decomposition for the Brownian snake (see [32] for Brownian excursions, [31] for Brownian snake or [1] for general homogeneous branching mechanism without spatial motion).

Under the excursion measures $\mathbf{N}_x^0$, $\tilde{\mathbf{N}}_x$ and $\mathbf{N}_x$, recall that $H_{\max} = \sup_{[0,\sigma]} H_s$. Because of the continuity of H, we can define $T_{\max} = \inf\{s>0, H_s = H_{\max}\}$. Notice the properties of the Brownian excursions implies that a.e. $H_s = H_{\max}$ only if $s = T_{\max}$. We set $v_t^0(x) = \mathbb{N}_x^0[H_{\max}>t]$ and recall this function does not depend on x. Thus, we shall write v_t^0 for $v_t^0(x)$. Standard computations give:

$$v_t^0 = \frac{\beta_0}{\mathrm{e}^{\beta_0 t}-1}\cdot$$

The next result is a straightforward adaptation from Theorem 3.3 of [1] and gives the distribution of $(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$ under $\mathbf{N}_x^0$.

Proposition 4.7 (Williams' decomposition under $\mathbf{N}_x^0$). *We have:*

(i) *The distribution of H_{max} under $\mathbf{N}_x^0$ is characterized by: $\mathbf{N}_x^0[H_{max}>h] = v_h^0$.*
(ii) *Conditionally on $\{H_{max}=h_0\}$, the law of $W_{T_{max}}$ under $\mathbf{N}_x^0$ is distributed as $Y_{[0,h_0)}$ under $\tilde{\mathrm{P}}_x$.*
(iii) *Conditionally on $\{H_{max}=h_0\}$ and $W_{T_{max}}$, $R^g_{T_{max}}$ and $R^d_{T_{max}}$ are under $\mathbf{N}_x^0$ independent Poisson point measures on $\mathbb{R}^+\times\bar{\Omega}$ with intensity:*

$$\mathbf{1}_{[0,h_0)}(s)ds\ \mathbf{1}_{\{H_{max}(W)<h_0-s\}}\ \mathbf{N}^0_{W_{T_{max}}(s)}[dW].$$

In other words, for any non-negative measurable function F, we have

$$\mathbf{N}_x^0\left[F(H_{max}, W_{T_{max}}, R^g_{T_{max}}, R^d_{T_{max}})\right] = -\int_0^\infty \partial_h v_h^0\ dh\ \tilde{\mathrm{E}}_x\left[F(h, Y_{[0,h)}, \hat{R}^{W,(h),g}, \hat{R}^{W,(h),d})\right],$$

where under $\tilde{\mathrm{E}}_x$ and conditionally on $Y_{[0,h)}$, $\hat{R}^{W,(h),g}$ and $\hat{R}^{W,(h),d}$ are two independent Poisson point measures with intensity $\hat{\nu}^{W,(h)}(ds,dW) = \mathbf{1}_{[0,h)}(s)ds\ \mathbf{1}_{\{H_{max}(W)<h-s\}}\ \mathbf{N}^0_{Y_s}[dW]$.

Notice that items (ii) and (iii) in the previous Proposition implies the existence of a measurable family $(\mathbf{N}_x^{0,(h)}, h>0)$ of probabilities on $(\bar{\Omega},\bar{\mathcal{G}})$ such that $\mathbf{N}_x^{0,(h)}$ is the distribution of W (more precisely of $(W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$) under $\mathbf{N}_x^0$ conditionally on $\{H_{\max}=h\}$.

Remark 4.8. In Klebaner & al [19], the Esty time reversal "is obtained by conditioning a [discrete time] Galton Watson process in negative time upon entering state 0 (extinction) at time 0 when starting at state 1 at time $-n$ and letting n tend to infinity". The authors then observe that in the linear fractional case (modified geometric offspring distribution) the Esty time reversal has the law of the same Galton Watson process conditioned on non extinction. Notice that in our continuous setting, the process $(H_s, 0 \leq s \leq T_{\max})$ is under $\mathbf{N}_x^{0,(h)}$ a Bessel process up to its first hitting time of h, and thus is reversible: $(H_s, 0 \leq s \leq T_{\max})$ under $\mathbf{N}_x^{0,(h)}$ is distributed as $(h - H_{T_{\max}-s}, 0 \leq s \leq T_{\max})$ under $\mathbf{N}_x^{0,(h)}$. It is also well known (see Corollary 3.1.6 of [12]) that $(H_{\sigma-s}, 0 \leq s \leq \sigma - T_{\max})$ under $\mathbf{N}_x^{0,(h)}$ is distributed as $(H_s, 0 \leq s \leq T_{\max})$ under $\mathbf{N}_x^{0,(h)}$. We deduce from these two points that $(X_s(1), 0 \leq s \leq h)$ under $\mathbf{N}_x^{0,(h)}$ is distributed as $(X_{h-s}(1), 0 \leq s \leq h)$ under $\mathbf{N}_x^{0,(h)}$. This result, which holds at fixed h, gives a pre-limiting version of the Esty time reversal in continuous time. Passing to the limit as $h \to \infty$, see Section 5.2, we get the equivalent of the Esty time reversal in a continuous setting.

Before stating the Williams' decomposition, Theorem 4.12, let us prove some properties for the functions $v_t(x) = \mathbb{N}_x[H_{\max} > t] = \mathbb{N}_x[X_t \neq 0]$ and $\tilde{v}_t(x) = \tilde{\mathbb{N}}_x[H_{\max} > t]$ which will play a significant rôle in the next Section. Recall (17) states that

$$\alpha v_t = \tilde{v}_t.$$

Notice also that (18) implies that q is bounded from above by $(\beta_0 + \| \tilde{\beta} \|_\infty)/2$.

Lemma 4.9. *Assume* $(H1)$-$(H3)$. *We have:*

$$q(x) + v_t^0 \geq \tilde{v}_t(x) \geq v_t^0. \tag{46}$$

Furthermore for fixed $x \in E$, $\tilde{v}_t(x)$ *is of class* $\mathcal{C}^1$ *in* t *and we have:*

$$\partial_t \tilde{v}_t(x) = \tilde{\mathrm{E}}_x \left[\mathrm{e}^{\int_0^t \Sigma_r(Y_{t-r})\, dr} \right] \partial_t v_t^0, \tag{47}$$

where the function Σ *defined by:*

$$\Sigma_t(x) = 2(v_t^0 + q(x) - \tilde{v}_t(x)) = \partial_\lambda \psi^0(v_t^0) - \partial_\lambda \tilde{\psi}(x, \tilde{v}_t(x)) \tag{48}$$

satisfies:

$$0 \leq \Sigma_t(x) \leq 2q(x) \leq \beta_0 + \| \tilde{\beta} \|_\infty . \tag{49}$$

Proof. We deduce from item (iii) of Proposition 3.7 that, as $\varphi \geq 0$ (see Lemma 3.8),

$$\tilde{v}_t(x) = \tilde{\mathbb{N}}_x[X_t \neq 0] = \mathbb{N}_x^0 \left[\mathbf{1}_{\{X_t \neq 0\}} \, \mathrm{e}^{\int_0^{+\infty} ds \; X_s(\varphi)} \right] \geq \mathbb{N}_x^0[X_t \neq 0] = v_t^0.$$

We also have

$$\begin{aligned} \tilde{v}_t(x) &= \mathbb{N}_x^0 \left[\mathbf{1}_{\{X_t \neq 0\}} \, \mathrm{e}^{\int_0^{+\infty} ds \; X_s(\varphi)} \right] \\ &= \mathbb{N}_x^0 \left[\mathrm{e}^{\int_0^{+\infty} ds \; X_s(\varphi)} - 1 \right] + \mathbb{N}_x^0 \left[1 - \mathbf{1}_{\{X_t = 0\}} \, \mathrm{e}^{\int_0^{+\infty} ds \; X_s(\varphi)} \right] \\ &= q(x) + \mathbb{N}_x^0 \left[1 - \mathbf{1}_{\{X_t = 0\}} \, \mathrm{e}^{\int_0^{+\infty} ds \; X_s(\varphi)} \right] \\ &\leq q(x) + \mathbb{N}_x^0 \left[1 - \mathbf{1}_{\{X_t = 0\}} \right] \\ &= q(x) + v_t^0, \end{aligned}$$

where we used (22) for the third equality. This proves (46).

Using the Williams' decomposition under $\mathbf{N}^0_x$, we get:

$$\tilde{v}_t(x) = -\int_t^{+\infty} \partial_r v^0_r \, dr \; \mathbf{N}^{0,(r)}_x \left[\mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right].$$

Using again the Williams' decomposition under $\mathbf{N}^0_x$, we have

$$\begin{aligned}\mathbf{N}^{0,(r)}_x \left[\mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right] &= \tilde{\mathrm{E}}_x \left[\mathrm{e}^{2\int_0^r ds\, \mathbf{N}^0_{Y_{r-s}} \left[(\mathrm{e}^{\int_0^{+\infty} dt\, X_t(\varphi)} - 1) \mathbf{1}_{\{X_s=0\}} \right]} \right] \\ &= \tilde{\mathrm{E}}_x \left[\mathrm{e}^{2\int_0^r ds\, \mathbf{N}^0_{Y_s} \left[(\mathrm{e}^{\int_0^{+\infty} dt\, X_t(\varphi)} - 1) \mathbf{1}_{\{X_{r-s}=0\}} \right]} \right]. \end{aligned} \tag{50}$$

We deduce that, for fixed x, $r \mapsto \mathbf{N}^{0,(r)}_x \left[\mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right]$ is non-decreasing and continuous as $\mathbf{N}^0_y[H_{\max} = t] = 0$ for $t > 0$. Therefore, we deduce that for fixed x, $\tilde{v}_t(x)$ is of class $\mathcal{C}^1$ in t:

$$\partial_t \tilde{v}_t(x) = \mathbf{N}^{0,(t)}_x \left[\mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right] \; \partial_t v^0_t.$$

We have thanks to item (iii) from Proposition 3.7:

$$\begin{aligned} &\mathbf{N}^0_y \left[(\mathrm{e}^{\int_0^{+\infty} dt\, X_t(\varphi)} - 1) \mathbf{1}_{\{X_s=0\}} \right] \\ &\quad = \mathbf{N}^0_y [X_s \neq 0] + \mathbf{N}^0_y \left[\mathrm{e}^{\int_0^{+\infty} dt\, X_t(\varphi)} - 1 \right] - \mathbf{N}^0_y \left[\mathrm{e}^{\int_0^{+\infty} dt\, X_t(\varphi)} \mathbf{1}_{\{X_s \neq 0\}} \right] \\ &\quad = v^0_s + q(y) - \tilde{v}_s(y) \\ &\quad = \frac{1}{2} \left[\partial_\lambda \psi^0(v^0_s) - \partial_\lambda \tilde{\psi}(y, \tilde{v}_s(y)) \right], \end{aligned} \tag{51}$$

where the last equality follows from (15), (18) and (19). Thus, with $\Sigma_s(y) = \partial_\lambda \psi^0(v^0_s) - \partial_\lambda \tilde{\psi}(y, \tilde{v}_s(y))$, we deduce that:

$$\mathbf{N}^{0,(t)}_x \left[\mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right] = \tilde{\mathrm{E}}_x \left[\mathrm{e}^{\int_0^t ds\, \Sigma_s(Y_{t-s})} \right].$$

This implies (47). Notice that, thanks to (46), Σ is non-negative and bounded from above by $2q$. $\square$

Fix $h > 0$. We define the probability measures $\mathrm{P}^{(h)}$ absolutely continuous with respect to P and $\tilde{\mathrm{P}}$ on $\mathcal{D}_h$ with Radon-Nikodym derivative:

$$\frac{d\mathrm{P}^{(h)}_{x\,|\mathcal{D}_h}}{d\tilde{\mathrm{P}}_{x\,|\mathcal{D}_h}} = \frac{\mathrm{e}^{\int_0^h \Sigma_{h-r}(Y_r)\, dr}}{\tilde{\mathrm{E}}_x \left[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\, dr} \right]}. \tag{52}$$

Notice this Radon-Nikodym derivative is 1 if the branching mechanism ψ is homogeneous. We deduce from (47) and (48) that:

$$\frac{d\mathrm{P}^{(h)}_{x\,|\mathcal{D}_h}}{d\tilde{\mathrm{P}}_{x\,|\mathcal{D}_h}} = \frac{\partial_h v^0_h}{\partial_h \tilde{v}_h(x)} \, \mathrm{e}^{-\int_0^h dr\, \left(\partial_\lambda \tilde{\psi}(Y_r, \tilde{v}_{h-r}(Y_r)) - \partial_\lambda \psi^0(v^0_{h-r}) \right)}$$

and, using (14):

$$\frac{d\mathrm{P}^{(h)}_{x\,|\mathcal{D}_h}}{d\mathrm{P}_{x\,|\mathcal{D}_h}} = \frac{1}{\alpha(Y_h)} \frac{\partial_h v^0_h}{\partial_h v_h(x)} \, \mathrm{e}^{-\int_0^h dr\, \left(\partial_\lambda \psi(Y_r, v_{h-r}(Y_r)) - \partial_\lambda \psi^0(v^0_{h-r}) \right)}. \tag{53}$$

In the next Lemma, we give an intrinsic representation of the Radon-Nikodym derivatives (52) and (53), which does not involve β_0 or v^0.

Lemma 4.10. *Assume (H1)-(H3). Fix $h > 0$. The processes $M^{(h)} = (M_t^{(h)}, t \in [0,h))$ and $\tilde{M}^{(h)} = (\tilde{M}_t^{(h)}, t \in [0,h))$, with:*

$$M_t^{(h)} = \frac{\partial_h v_{h-t}(Y_t)}{\partial_h v_h(x)} \, \mathrm{e}^{-\int_0^t ds \; \partial_\lambda \psi(Y_s, v_{h-s}(Y_s))} \quad \text{and} \quad \tilde{M}_t^{(h)} = \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)} \, \mathrm{e}^{-\int_0^t ds \; \partial_\lambda \tilde{\psi}(Y_s, \tilde{v}_{h-s}(Y_s))},$$

are non-negative bounded $\mathcal{D}_t$-martingales respectively under P_x and $\tilde{\mathrm{P}}_x$. Furthermore, we have for $0 \leq t < h$:

$$\frac{d\mathrm{P}^{(h)}_{x\;|\mathcal{D}_t}}{d\mathrm{P}_{x\;|\mathcal{D}_t}} = M_t^{(h)} \quad \text{and} \quad \frac{d\mathrm{P}^{(h)}_{x\;|\mathcal{D}_t}}{d\tilde{\mathrm{P}}_{x\;|\mathcal{D}_t}} = \tilde{M}_t^{(h)}. \tag{54}$$

Notice the limit $M_h^{(h)}$ of $M^{(h)}$ and the limit $\tilde{M}_h^{(h)}$ of $\tilde{M}^{(h)}$ are respectively given by the right-handside of (53) and (52).

Remark 4.11. Comparing (10) and (54), we have that $\mathrm{P}_x^{(h)} = \mathrm{P}_x^g$ with $g(t,x) = \partial_h v_{h-t}(x)$, if g satisfies the assumptions of Remark 3.3.

Proof. First of all, the process $\tilde{M}^{(h)}$ is clearly $\mathcal{D}_t$-adapted. Using (47), we get:

$$\tilde{\mathrm{E}}_y \left[\mathrm{e}^{\int_0^{h-t} \Sigma_{h-t-r}(Y_r)\, dr} \right] = \frac{\partial_h \tilde{v}_{h-t}(y)}{\partial_h v^0_{h-t}}.$$

We set:

$$\tilde{M}_h^{(h)} = \frac{\mathrm{e}^{\int_0^h \Sigma_{h-r}(Y_r)\, dr}}{\tilde{\mathrm{E}}_x \left[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\, dr} \right]}.$$

We have:

$$\begin{aligned}
\tilde{\mathrm{E}}_x[\tilde{M}_h^{(h)} | \mathcal{D}_t] &= \frac{\mathrm{e}^{\int_0^t \Sigma_{h-r}(Y_r)\, dr}}{\tilde{\mathrm{E}}_x \left[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\, dr} \right]} \tilde{\mathrm{E}}_{Y_t} \left[\mathrm{e}^{\int_0^{h-t} \Sigma_{h-t-r}(Y_r)\, dr} \right] \\
&= \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)} \frac{\partial_h v^0_h}{\partial_h v^0_{h-t}} \, \mathrm{e}^{\int_0^t \Sigma_{h-r}(Y_r)\, dr} \\
&= \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)} \, \mathrm{e}^{-\int_0^t \partial_\lambda \tilde{\psi}\left(Y_s, \tilde{v}_{h-s}(Y_s)\right)\, ds} \frac{\partial_h v^0_h}{\partial_h v^0_{h-t}} \, \mathrm{e}^{\int_0^t \partial_\lambda \psi^0\left(v^0_{h-s}\right)\, ds}
\end{aligned}$$

In the homogeneous setting, v^0 simply solves the ordinary differential equation:

$$\partial_h v^0_h = -\psi^0(v^0_h).$$

This implies that

$$\partial_h \log(\partial_h v^0_h) = \frac{\partial_h^2 v^0_h}{\partial_h v^0_h} = -\partial_\lambda \psi^0(v^0_h)$$

and thus

$$\frac{\partial_h v^0_h}{\partial_h v^0_{h-t}} \, \mathrm{e}^{\int_0^t \partial_\lambda \psi^0\left(v^0_{h-s}\right)\, ds} = 1. \tag{55}$$

We deduce that

$$\tilde{\mathrm{E}}_x[\tilde{M}_h^{(h)} | \mathcal{D}_t] = \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)} \, \mathrm{e}^{-\int_0^t dr \; \partial_\lambda \tilde{\psi}(Y_r, \tilde{v}_{h-r}(Y_r))} = \tilde{M}_t^{(h)}.$$

Therefore, $\tilde{M}^{(h)}$ is a $\mathcal{D}_t$-martingale under $\tilde{\mathrm{P}}_x$ and the second part of (54) is a consequence of (52). Then, use (14) to get that $M^{(h)}$ is a $\mathcal{D}_t$-martingale under P_x and the first part of (54). $\square$

We now give the Williams' decomposition: the distribution of $(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$ under $\mathbf{N}_x$ or equivalently under $\tilde{\mathbf{N}}_x/\alpha(x)$. Recall the distribution $\mathrm{P}^{(h)}_x$ defined in (52) or (53).

Theorem 4.12 (Williams' decomposition under $\mathbf{N}_x$)**.** *Assume* $(H1)$-$(H3)$. *We have:*

(i) *The distribution of H_{max} under $\mathbf{N}_x$ is characterized by: $\mathbf{N}_x[H_{max} > h] = v_h(x)$.*
(ii) *Conditionally on $\{H_{max} = h_0\}$, the law of $W_{T_{max}}$ under $\mathbf{N}_x$ is distributed as $Y_{[0,h_0)}$ under $\mathrm{P}^{(h_0)}_x$.*
(iii) *Conditionally on $\{H_{max} = h_0\}$ and $W_{T_{max}}$, $R^g_{T_{max}}$ and $R^d_{T_{max}}$ are under $\mathbf{N}_x$ independent Poisson point measures on $\mathbb{R}^+ \times \bar{\Omega}$ with intensity:*

$$\mathbf{1}_{[0,h_0)}(s)ds\; \mathbf{1}_{\{H_{max}(W')<h_0-s\}}\alpha(W_{T_{max}}(s))\; \mathbf{N}_{W_{T_{max}}(s)}[dW'].$$

In other words, for any non-negative measurable function F, we have

$$\mathbf{N}_x\Big[F(H_{max}, W_{T_{max}}, R^g_{T_{max}}, R^d_{T_{max}})\Big] = -\int_0^\infty \partial_h v_h(x)\; dh\; \mathrm{E}^{(h)}_x\Big[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\Big],$$

where under $\mathrm{E}^{(h)}_x$ and conditionally on $Y_{[0,h)}$, $R^{W,(h),g}$ and $R^{W,(h),d}$ are two independent Poisson point measures with intensity:

$$\nu^{W,(h)}(ds, dW) = \mathbf{1}_{[0,h)}(s)ds\; \mathbf{1}_{\{H_{max}(W)<h-s\}}\alpha(Y_s)\; \mathbf{N}_{Y_s}[dW]. \tag{56}$$

Notice that items (ii) and (iii) in the previous Proposition imply the existence of a measurable family $(\mathbf{N}^{(h)}_x, h > 0)$ of probabilities on $(\bar{\Omega}, \bar{\mathcal{G}})$ such that $\mathbf{N}^{(h)}_x$ is the distribution of W (more precisely of $(W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$) under $\mathbf{N}_x$ conditionally on $\{H_{\max} = h\}$.

Proof. We keep notations introduced in Proposition 4.7 and Theorem 4.12. We have:

$$\begin{aligned}
&\tilde{\mathbf{N}}_x\Big[F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\Big]\\
&= \mathbf{N}^0_x\Big[\mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)}\, F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\Big]\\
&= \mathbf{N}^0_x\Big[F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\, \mathrm{e}^{(R^g_{T_{\max}}+R^d_{T_{\max}})(f)}\Big]\\
&= -\int_0^\infty \partial_h v^0_h\; dh\; \tilde{\mathrm{E}}_x\Big[F(h, Y_{[0,h)}, \hat{R}^{W,(h),g}, \hat{R}^{W,(h),d})\, \mathrm{e}^{(\hat{R}^{W,(h),g}+\hat{R}^{W,(h),d})(f)}\Big]\\
&= -\int_0^\infty \partial_h v^0_h\; dh\; \tilde{\mathrm{E}}_x\Big[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\, \mathrm{e}^{2\int_0^h ds\, \mathbf{N}^0_{Y_s}\Big[(\mathrm{e}^{\int_0^{+\infty} dt\, X_t(\varphi)}-1)\mathbf{1}_{\{X_{r-s}=0\}}\Big]}\Big]\\
&= -\int_0^\infty \partial_h v^0_h\; dh\; \tilde{\mathrm{E}}_x\Big[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\, \mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\, ds}\Big]\\
&= -\int_0^\infty \partial_h v^0_h\; dh\; \tilde{\mathrm{E}}_x\Big[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\, ds}\Big]\mathrm{E}^{(h)}_x\Big[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\Big]\\
&= -\int_0^\infty \partial_h \tilde{v}_h(x)\; dh\; \mathrm{E}^{(h)}_x\Big[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\Big],
\end{aligned}$$

where the first equality comes from $(H1)$ and item (iii) of Proposition 3.7; we set $f(s, W) = \int_0^{+\infty} X_r(W)(\varphi)$ for the second equality; we use Proposition 4.7 for the third equality; we use

Lemma 4.3 for the fourth with $R^{W,(h),g}$ and $R^{W,(h),d}$ which under $\tilde{\mathrm{E}}_x^{(h)}$ and conditionally on $Y_{[0,h)}$ are two independent Poisson point measures with intensity $\nu^{W,(h)}$; we use (51) for the fifth, definition (52) of $\mathrm{E}_x^{(h)}$ for the sixth, and (47) for the seventh. Then use (33) and (17) to conclude. □

The definition of $\mathbf{N}_x^{(h)}$ gives in turn sense to the conditional law $\mathbb{N}_x^{(h)} = \mathbb{N}_x(.|H_{\max} = h)$ of the $(\mathcal{L}, \beta, \alpha)$ superprocess conditioned to die at time h, for all $h > 0$. The next Corollary is then a straightforward consequence of Theorem 4.12.

Corollary 4.13. *Assume (H1)-(H3). Let $h > 0$. Let $x \in E$ and $Y_{[0,h)}$ be distributed according to $\mathrm{P}_x^{(h)}$. Consider the Poisson point measure $\mathcal{N} = \sum_{j\in J} \delta_{(s_j, X^j)}$ on $[0, h) \times \Omega$ with intensity:*

$$2\mathbf{1}_{[0,h)}(s)ds\ \mathbf{1}_{\{H_{max}(X)<h-s\}}\alpha(Y_s)\ \mathbb{N}_{Y_s}[dX].$$

The process $X^{(h)} = (X_t^{(h)}, t \geq 0)$, which is defined for all $t \geq 0$ by:

$$X_t^{(h)} = \sum_{j\in J,\, s_j<t} X^j_{t-s_j},$$

is distributed according to $\mathbb{N}_x^{(h)}$.

We now give the superprocess counterpart of Theorem 4.12.

Corollary 4.14 (Williams' decomposition under $\mathbb{P}_\nu$)**.** *Assume (H1)-(H3). We have the following result.*

(i) *Sample a positive number h_0 according to the law of H_{max} under $\mathbb{P}_\nu$: $\mathbb{P}_\nu(H_{max} \leq h) = \mathrm{e}^{-\nu(v_h)}$.*
(ii) *Conditionally on h_0, sample $x_0 \in E$ according to the probability measure*

$$\frac{\partial_h v_{h_0}(x)}{\nu(\partial_h v_{h_0})}\nu(dx).$$

(iii) *Conditionally on h_0 and x_0, sample $X^{(h_0)}$ according to the probability measure $\mathbb{N}_{x_0}^{(h_0)}$.*
(iv) *Conditionally on h_0, sample X', independent of x_0 and $X^{(h_0)}$, according to the probability measure $\mathbb{P}_\nu(.|H_{max} < h_0)$.*

Then the measure valued process $X' + X^{(h_0)}$ has distribution $\mathbb{P}_\nu$.

In particular the distribution of $X'+X^{(h_0)}$ conditionally on h_0 (which is given by (ii)-(iv) from Corollary 4.14) is a regular version of the distribution of the $(\mathcal{L}, \beta, \alpha)$ superprocess conditioned to die at a fixed time h_0, which we shall write $\mathbb{P}_\nu^{(h_0)}$.

Proof. Let μ be a finite measure on $\mathbb{R}^+$ and f a non-negative measurable function defined on $\mathbb{R}^+ \times E$. For a measure-valued process $Z = (Z_t, t \geq 0)$ on E, we set $Z(f\mu) = \int f(t, x)\ Z_t(dx)\mu(dt)$. We also write $f_s(t, x) = f(s + t, x)$.

Let X' and $X^{(h_0)}$ be defined as in Corollary 4.14. In order to characterized the distribution of the process $X' + X^{(h_0)}$, we shall compute

$$A = \mathbb{E}[\mathrm{e}^{-X'(f\mu)-X^{(h_0)}(f\mu)}].$$

We shall use notations from Corollary 4.13. We have:

$$\begin{aligned}
A &= -\int_0^{+\infty} \nu(\partial_h v_h)\, \mathrm{e}^{-\nu(h)}\, dh \int_E \frac{\partial_h v_h(x)}{\nu(\partial_h v_h)}\, \nu(dx) \\
&\qquad\qquad \mathrm{E}_x^{(h)}\left[\mathbb{E}[\mathrm{e}^{-\sum_{j\in J} X^j(f_{s_j}\mu)}\,|Y_{[0,h)}]\right] \mathbb{E}_\nu\left[\mathrm{e}^{-X(f\mu)}\,|H_{\max}<h\right] \\
&= -\int_E \nu(dx) \int_0^{+\infty} \partial_h v_h(x)\, dh \\
&\qquad\qquad \mathrm{E}_x^{(h)}\left[\mathbb{E}[\mathrm{e}^{-\sum_{j\in J} X^j(f_{s_j}\mu)}\,|Y_{[0,h)}]\right] \mathbb{E}_\nu\left[\mathrm{e}^{-X(f\mu)}\,\mathbf{1}_{\{H_{\max}<h\}}\right],
\end{aligned}$$

where we used the definition of X' and $\mathcal{N}$ for the first equality, and the equality $\mathbb{P}_\nu(H_{\max} < h) = \mathbb{P}_\nu(H_{\max} \leq h) = \mathrm{e}^{-\nu(h)}$ for the second. Recall notations from Theorem 4.12. We set:

$$G\left(\sum_{i\in I}\delta_{(s_i,W^i)}, \sum_{i'\in I'}\delta_{(s_{i'},W^{i'})}\right) = \mathrm{e}^{-\sum_{j\in I\cup I'} X^j(W^j)(f_{s_j}\mu)}$$

and $g(h) = \mathbb{E}_\nu\left[\mathrm{e}^{-X(f\mu)}\,\mathbf{1}_{\{H_{\max}<h\}}\right]$. We have:

$$\begin{aligned}
A &= -\int_E \nu(dx) \int_0^{+\infty} \partial_h v_h(x)\, dh\, \mathrm{E}_x^{(h)}\left[G(R^{W,(h),g}, R^{W,(h),d})g(h)\right] \\
&= \int_E \nu(dx)\, \mathbf{N}_x\left[G(R_g^{T_{\max}}, R_d^{T_{\max}})g(H_{\max})\right] \\
&= \int_E \nu(dx)\, \mathbb{N}_x\left[\mathrm{e}^{-X(f\mu)}\, \mathbb{E}_\nu\left[\mathrm{e}^{-X(f\mu)}\,\mathbf{1}_{\{H_{\max}<h\}}\right]_{|h=H_{\max}}\right] \\
&= \mathbb{E}\left[\sum_{i\in I}\mathrm{e}^{-X^i(f\mu)} \prod_{j\in I;\, j\neq i} \mathrm{e}^{-X^j(f\mu)}\,\mathbf{1}_{\{H^j_{\max}<H^i_{\max}\}}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{-\sum_{i\in I} X^i(f\mu)}\right] \\
&= \mathbb{E}_\nu\left[\mathrm{e}^{-X(f\mu)}\right],
\end{aligned}$$

where we used the definition of G and g for the first and third equalities, Theorem 4.12 for the second equality, the master formula for Poisson point measure $\sum_{i\in I}\delta_{X^i}$ with intensity $\nu(dx)\,\mathbb{N}_x[dX]$ for the fourth equality (and the obvious notation $H^i_{\max} = \inf\{t\geq 0; X^i_t = 0\}$) and Theorem 2.3 for the last equality. Thus we get:

$$\mathbb{E}[\mathrm{e}^{-X'(f\mu)-X^{(h_0)}(f\mu)}] = \mathbb{E}_\nu\left[\mathrm{e}^{-X(f\mu)}\right].$$

This readily implies that the process $X' + X^{(h_0)}$ is distributed as X under $\mathbb{P}_\nu$. □

5. Some applications

5.1. **The law of the Q-process.** Recall $\mathbb{P}_\nu^{(h)}$ defined after Corollary 4.14 is the distribution of the $(\mathcal{L},\beta,\alpha)$-superprocess started at $\nu\in\mathcal{M}_f(E)$ conditionally on $\{H_{\max}=h\}$. We consider also $\mathbb{P}_\nu^{(\geq h)} = \mathbb{P}_\nu(\,\cdot\,|H_{\max}\geq h)$ and $\mathbb{N}_x^{(\geq h)} = \mathbb{N}_x(\,\cdot\,|H_{\max}\geq h)$ the distributions of the $(\mathcal{L},\beta,\alpha)$-superprocess conditionally on $\{H_{\max}\geq h\}$.

The distribution of the Q-process, when it exists, is defined as the weak limit of $\mathbb{P}_\nu^{(\geq h)}$ when h goes to infinity. The next Lemma insures that if $\mathbb{P}_\nu^{(h)}$ weakly converges to a limit $\mathbb{P}_\nu^{(\infty)}$, then this limit is also the distribution of the Q-process.

Lemma 5.1. *Fix $t > 0$. If $\mathbb{P}_\nu^{(h)}$ (resp. $\mathbb{N}_x^{(h)}$) converges weakly to $\mathbb{P}_\nu^{(\infty)}$ (resp. $\mathbb{N}_x^{(\infty)}$) on $(\Omega, \mathcal{F}_t)$, then $\mathbb{P}_\nu^{(\geq h)}$ (resp. $\mathbb{N}_x^{(\geq h)}$) converges weakly to $\mathbb{P}_\nu^{(\infty)}$ (resp. $\mathbb{N}_x^{(\infty)}$) on $(\Omega, \mathcal{F}_t)$.*

Proof. Let $Z = \mathbf{1}_A$ with $A \in \mathcal{F}_t$ such that $\mathbb{P}_\nu^{(\infty)}(\partial A) = 0$. Using the Williams' decomposition under $\mathbb{P}_\nu$ given by Corollary 4.14, we have for $h > t$:

$$\mathbb{E}_\nu^{(\geq h)}[Z] = \mathrm{e}^{\nu(v_h)} \int_h^\infty \mathbb{E}_\nu^{(h')}[Z]\, f(h')dh',$$

where $f(h) = -\nu(\partial_h v_h)\exp(-\nu(v_h))$. We write down the difference:

$$\mathbb{E}_\nu^{(\geq h)}[Z] - \mathbb{E}_\nu^{(\infty)}[Z] = \mathrm{e}^{\nu(v_h)} \int_h^\infty \left(\mathbb{E}_\nu^{(h')}[Z] - \mathbb{E}_\nu^{(\infty)}[Z]\right) f(h')dh'.$$

Since $\mathbb{P}_\nu^{(h')}$ weakly converges to $\mathbb{P}_\nu^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$ and since $\mathbb{P}_\nu^{(\infty)}(\partial A) = 0$, we deduce that $\lim_{h'\to+\infty} \mathbb{E}_\nu^{(h')}[Z] - \mathbb{E}_\nu^{(\infty)}[Z] = 0$. We conclude that $\lim_{h\to+\infty} \mathbb{E}_\nu^{(\geq h)}[Z] - \mathbb{E}_\nu^{(\infty)}[Z] = 0$, which gives the result. The proof is similar for the conditioned excursion measures. □

We now address the question of convergence of the family of probability measures $(\mathbb{P}_x^{(h)}, h \geq 0)$. Recall from (54) that for all $0 \leq t < h$:

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = M_t^{(h)}.$$

We shall consider the following assumption on the convergence in law of the spine.

$(H4)$ **For all $t \geq 0$, P_x-a.s. $(M_t^{(h)}, h > t)$ converges to a limit say $M_t^{(\infty)}$, and $\mathrm{E}_x[M_t^{(\infty)}] = 1$.**

Note that Scheffé's lemma implies that the convergence also holds in $\mathrm{L}^1(\mathrm{P}_x)$. Furthermore, since $(M_t^{(h)}, t \in [0, h))$ is a non-negative martingale, there exists a version of $(M_t^{(\infty)}, t \geq 0)$ which is a non-negative martingale.

Remark 5.2. We provide in Section 7 sufficient conditions for $(H1)$-$(H4)$ to hold in the case of the multitype Feller diffusion and the superdiffusion. These conditions are stated in term of the generalized eigenvalue λ_0 defined by

$$\lambda_0 = \sup\{\ell \in \mathbb{R}, \exists u \in \mathcal{D}(\mathcal{L}), u > 0 \text{ such that } (\beta - \mathcal{L})u = \ell\ u\}, \tag{57}$$

and its associated eigenfunction.

Remark 5.3. The family $(\mathrm{P}_x^{(h)}, h \geq 0)$ and the family $(\mathrm{P}_x^{(B,h)}, h \geq 0)$ defined in Remark 4.6 will be shown in Lemma 6.13 to converge to the same limiting probability measure.

Under $(H4)$, we define the probability measure $\mathrm{P}_x^{(\infty)}$ on $(D, \mathcal{D})$ by its Radon Nikodym derivative, for all $t \geq 0$:

$$\frac{d\mathrm{P}^{(\infty)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = M_t^{(\infty)}. \tag{58}$$

By construction, the probability measure $\mathrm{P}_x^{(h)}$ converges weakly to $\mathrm{P}_x^{(\infty)}$ on $\mathcal{D}_t$, for all $t \geq 0$.

Let $\nu \in \mathcal{M}_f(E)$. We shall consider the following assumption:

$(H5)_\nu$ **There exists a measurable function ρ such that the following convergence holds in $\mathrm{L}^1(\nu)$:**

$$\frac{\partial_h v_h}{\nu(\partial_h v_h)} \xrightarrow[h\to+\infty]{} \rho.$$

In particular, we have $\nu(\rho)=1$. Let $\nu \in \mathcal{M}_f(E)$. Under $(H4)$ and $(H5)_\nu$, we set:

$$\mathrm{P}_\nu^{(\infty)}(dY) = \int_E \nu(dx)\rho(x)\, \mathrm{P}_x^{(\infty)}(dY).$$

Notice then that $\int_E \nu(dx)\, \frac{\partial_h v_h(x)}{\nu(\partial_h v_h)} \mathrm{P}_x^{(h)}(dY)$ converges weakly to $\mathrm{P}_\nu^{(\infty)}(dY)$ on $\mathcal{D}_t$, for all $t \geq 0$.

Remark 5.4. If ν a constant times the Dirac mass δ_x, for some $x \in E$, then $(H5)_\nu$ holds if $(H4)$ holds and in this case we have $\mathrm{P}_\nu^{(\infty)} = \mathrm{P}_x^{(\infty)}$.

We can now state the result on the convergence of $\mathbf{N}_x^{(h)}$.

Theorem 5.5. *Assume $(H1)$-$(H4)$. Let $t \geq 0$. The triplet $((W_{T_{max}})_{[0,t]}, (R^g_{T_{max}})_{[0,t]}, (R^d_{T_{max}})_{[0,t]})$ under $\mathbf{N}_x^{(h)}$ converges weakly to the distribution of the triplet $(Y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]})$ where Y has distribution $\mathrm{P}_x^{(\infty)}$ and conditionally on Y, $R^{B,g}$ and $R^{B,d}$ are two independent Poisson point measures with intensity ν^B given by (39). We even have the slightly stronger result. For any bounded measurable function F, we have:*

$$\mathbf{N}_x^{(h)}\Big[F\big((W_{T_{max}})_{[0,t]}, (R^g_{T_{max}})_{[0,t]}, (R^d_{T_{max}})_{[0,t]}\big)\Big] \xrightarrow[h\to+\infty]{} \mathrm{E}_x^{(\infty)}\Big[F\big(Y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]}\big)\Big]. \tag{59}$$

Proof. Let $h > t$. We use notations from Theorems 5.5 and 4.12. Let F be a bounded measurable function on $\mathcal{W} \times (\mathbb{R}^+ \times \bar{\Omega})^2$. From the Williams' decomposition, Theorem 4.12, we have:

$$\begin{aligned}\mathbf{N}_x^{(h)}\Big[F((W_{T_{\max}})_{[0,t]}, (R^g_{T_{\max}})_{[0,t]}, (R^d_{T_{\max}})_{[0,t]})\Big] &= \mathrm{E}_x^{(h)}\Big[F\big(Y_{[0,t]}, R^{W,g,(h)}_{[0,t]}, R^{W,d,(h)}_{[0,t]}\big)\Big]\\ &= \mathrm{E}_x^{(h)}\big[\varphi^h(Y_{[0,t]})\big],\end{aligned}$$

where φ^h is defined by:

$$\varphi^h(y_{[0,t]}) = \mathrm{E}_x^{(h)}\Big[F\big(y_{[0,t]}, R^{W,g,(h)}_{[0,t]}, R^{W,d,(h)}_{[0,t]}\big)\Big| Y=y\Big].$$

We also set:

$$\varphi^\infty(y_{[0,t]}) = \mathrm{E}_x^{(\infty)}\Big[F\big(y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]}\big)\Big| Y=y\Big].$$

We want to control:

$$\Delta_h = \mathbf{N}_x^{(h)}\Big[F((W_{T_{\max}})_{[0,t]}, (R^g_{T_{\max}})_{[0,t]}, (R^d_{T_{\max}})_{[0,t]})\Big] - \mathrm{E}_x^{(\infty)}\Big[F(Y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]})\Big].$$

Notice that:

$$\begin{aligned}\Delta_h &= \mathrm{E}_x^{(h)}\big[\varphi^h(Y_{[0,t]})\big] - \mathrm{E}_x^{(\infty)}\big[\varphi^\infty(Y_{[0,t]})\big]\\ &= \big(\mathrm{E}_x^{(h)}\big[\varphi^h(Y_{[0,t]})\big] - \mathrm{E}_x^{(\infty)}\big[\varphi^h(Y_{[0,t]})\big]\big) + \mathrm{E}_x^{(\infty)}\big[(\varphi^h - \varphi^\infty)(Y_{[0,t]})\big].\end{aligned} \tag{60}$$

We prove the first term of the right hand-side of (60) converges to 0. We have:

$$\mathrm{E}_x^{(h)}\big[\varphi^h(Y_{[0,t]})\big] - \mathrm{E}_x^{(\infty)}\big[\varphi^h(Y_{[0,t]})\big] = \mathrm{E}_x^{(\infty)}\big[(M_t^{(h)} - M_t^{(\infty)})\, \varphi^h(Y_{[0,t]})\big].$$

Then use that φ^h is bounded by $\| F \|_\infty$ and the convergence of $(M_t^{(h)}, h > t)$ towards $M_t^{(\infty)}$ in $\mathrm{L}^1(\mathrm{P}_x^{(\infty)})$ to get:

$$\lim_{h\to\infty} \mathrm{E}_x^{(h)}\big[\varphi^h(Y_{[0,t]})\big] - \mathrm{E}_x^{(\infty)}\big[\varphi^h(Y_{[0,t]})\big] = 0. \tag{61}$$

We then prove the second term of the right hand-side of (60) converges to 0. Conditionally on Y, $R_{[0,t]}^{W,g,(h)}$ and $R_{[0,t]}^{W,d,(h)}$ (resp. $R_{[0,t]}^{B,g}$ and $R_{[0,t]}^{B,d}$) are independent Poisson point measures with intensity $\mathbf{1}_{[0,t]}(s)\ \nu^{W,(h)}(ds, dW)$ where $\nu^{W,(h)}$ is given by (56) (resp. $\mathbf{1}_{[0,t]}(s)\ \nu^B(ds, dW)$ where ν^B is given by (39)). And we have:

$$\mathbf{1}_{[0,t]}(s)\ \nu^{W,(h)}(ds, dW) = \mathbf{1}_{\{H_{\max}(W)<h-s\}}\mathbf{1}_{[0,t]}(s)\ \nu^B(ds, dW).$$

Thanks to (17) and (46), we get that:

$$\int \mathbf{1}_{\{H_{\max}(W)\geq h-s\}}\mathbf{1}_{[0,t]}(s)\ \nu^B(ds, dW) = \int_0^t ds\ \alpha(y_s)\mathbb{N}_{y_s}[H_{\max} \geq h-s] = \int_0^t ds\ v_{h-s}(y_s) < +\infty.$$

The proof of the next Lemma is postponed to the end of this Section.

Lemma 5.6. *Let R and $\tilde{R}$ be two Poisson point measures on a Polish space with respective intensity ν and $\tilde{\nu}$. Assume that $\tilde{\nu}(dx) = \mathbf{1}_A(x)\nu(dx)$, where A is measurable and $\nu(A^c) < +\infty$. Then for any bounded measurable function F, we have:*

$$\Big|\mathbb{E}[F(R)] - \mathbb{E}[F(\tilde{R})]\Big| \leq 2 \| F \|_\infty \nu(A^c).$$

Using this Lemma with ν given by $\mathbf{1}_{[0,t]}(s)\ \nu^B(ds, dW)$ and A given by $\{H_{\max}(W) < h - s\}$, we deduce that:

$$\Big|(\varphi^h - \varphi^\infty)(y_{[0,t]})\Big| \leq 4 \| F \|_\infty \int_0^t ds\ v_{h-s}(y_s).$$

We deduce that:

$$\Big|\mathrm{E}_x^{(\infty)}\big[(\varphi^h - \varphi^\infty)(Y_{[0,t]})\big]\Big| \leq 4 \| F \|_\infty \mathrm{E}_x^{(\infty)}\left[\int_0^t ds\ v_{h-s}(Y_s)\right].$$

Recall that $(H1)$ implies that $v_{h-s}(x)$ converges to 0 as h goes to infinity. Since v is bounded (use (17) and (46)), by dominated convergence, we get:

$$\lim_{h\to\infty} \mathrm{E}_x^{(\infty)}\big[(\varphi^h - \varphi^\infty)(Y_{[0,t]})\big] = 0. \tag{62}$$

Therefore, we deduce from (60) that $\lim_{h\to+\infty} \Delta_h = 0$, which gives (59). □

We now define a superprocess with spine distribution $\mathrm{P}_\nu^{(\infty)}$.

Definition 5.7. *Let $\nu \in \mathcal{M}_f(E)$. Assume $\mathrm{P}_\nu^{(\infty)}$ is well defined. Let Y be distributed according to $\mathrm{P}_\nu^{(\infty)}$, and, conditionally on Y, let $\mathcal{N} = \sum_{j\in J} \delta_{(s_j, X^j)}$ be a Poisson point measure with intensity:*

$$2\mathbf{1}_{\mathbb{R}^+}(s)ds\ \alpha(Y_s)\mathbb{N}_{Y_s}[dX].$$

Consider the process $X^{(\infty)} = (X_t^{(\infty)}, t \geq 0)$, which is defined for all $t \geq 0$ by:

$$X_t^{(\infty)} = \sum_{j\in J,\, s_j<t} X^j_{t-s_j}.$$

(i) *Let X' independent of $X^{(\infty)}$ and distributed according to $\mathbb{P}_\nu$. Then, we write $\mathbb{P}_\nu^{(\infty)}$ for the distribution of $X' + X^{(\infty)}$.*

(ii) *If ν is the Dirac mass at x, we write $\mathbb{N}_x^{(\infty)}$ for the distribution of $X^{(\infty)}$.*

As a consequence of Theorem 5.5, we get the convergence of $\mathbb{P}_\nu^{(h)}$. We shall write $\mathbb{P}_x^{(h)}$ when ν is the Dirac mass at x.

Corollary 5.8. *Under $(H1)$-$(H4)$, we have that, for all $t \geq 0$:*

(i) *The distribution $\mathbb{N}_x^{(h)}$ converges weakly to $\mathbb{N}_x^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$.*
(ii) *The distribution $\mathbb{P}_x^{(h)}$ converges weakly to $\mathbb{P}_x^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$.*
(iii) *Let $\nu \in \mathcal{M}_f(E)$. If furthermore $(H5)_\nu$ holds, then the distribution $\mathbb{P}_\nu^{(h)}$ converges weakly to $\mathbb{P}_\nu^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$.*

Proof. Point (i) is a direct consequence of Theorem 5.5, Definition 5.7 and Proposition 3.12.

Point (ii) is a direct consequence of point (i), Corollary 4.14 and the weak convergence of $\mathbb{P}_x^{(\leq h)}$ to $\mathbb{P}_x$ as h goes to infinity.

According to Corollary 4.14, under $\mathbb{P}_\nu^{(h)}$, X is distributed according to $X' + X^{(h)}$ where X' and $X^{(h)}$ are independent, X' is distributed according to $\mathbb{P}_\nu^{(\leq h)}$ and $X^{(h)}$ is distributed according to

$$\int_E \nu(dx)\, \frac{\partial_h v_h(x)}{\nu(\partial_h v_h)}\, \mathbb{N}_x^{(h)}[dX].$$

Assumption $(H5)_\nu$ implies this distribution converges weakly to:

$$\int_E \nu(dx)\, \rho(x)\, \mathbb{N}_x^{(\infty)}[dX]$$

(because of the convergence of the densities in $\mathrm{L}^1(\nu)$) on $(\Omega, \mathcal{F}_t)$ as h goes to infinity. This and the weak convergence of $\mathbb{P}_\nu^{(\leq h)}$ to $\mathbb{P}_\nu$ as h goes to infinity gives point (iii). $\square$

Proof of Lemma 5.6. Similarly to Lemma 4.3 (formally take $f = -\infty \mathbf{1}_{A^c}$), we have:

$$\mathbb{E}\left[F(R)\mathbf{1}_{\{R(A^c)=0\}}\right] = \mathbb{E}\left[F(\tilde{R})\right] \mathrm{e}^{-\nu(A^c)}.$$

We deduce that:

$$\begin{aligned}
\left|\mathbb{E}[F(R)] - \mathbb{E}[F(\tilde{R})]\right| &= \left|\mathbb{E}[F(R)] - \mathbb{E}[F(R)\mathbf{1}_{\{R(A^c)=0\}}]\,\mathrm{e}^{\nu(A^c)}\right| \\
&\leq \left|\mathbb{E}[F(R)] - \mathbb{E}[F(R)\mathbf{1}_{\{R(A^c)=0\}}]\right| + \left|\mathbb{E}[F(R)\mathbf{1}_{\{R(A^c)=0\}}](1 - \mathrm{e}^{\nu(A^c)})\right| \\
&\leq \| F \|_\infty (1 - \mathbb{P}(R(A^c) = 0)) + \| F \|_\infty\, \mathbb{P}(R(A^c) = 0)(\mathrm{e}^{\nu(A^c)} - 1) \\
&= 2 \| F \|_\infty (1 - \mathrm{e}^{-\nu(A^c)}) \\
&\leq 2 \| F \|_\infty\, \nu(A^c).
\end{aligned}$$

This gives the result. $\square$

5.2. **Backward from the extinction time.** We shall work in this section with the space $D^- = D(\mathbb{R}^-, E)$ equipped with the Skorokhod topology. We also consider the σ-fields $\mathcal{D}_I = \sigma(Y_r, r \in I)$ for I an interval on $(-\infty, 0]$.

Let us denote by θ the translation operator, which maps any process R to the shifted process $\theta_h(R)$ defined by:

$$\theta_h(R)_\cdot = R_{\cdot + h}\ .$$

The process R may be a path, a killed path or a point measure, in which case we set, for $R = \sum_{j\in J} \delta_{(s_j,x_j)}$, $\theta_h(R) = \sum_{j\in J} \delta_{(h+s_j,x_j)}$. We also denote $\mathrm{P}^{(-h)}$ the push forward probability measure of $\mathrm{P}^{(h)}$ by θ_h, defined on $\mathcal{D}_{[-h,0]}$ by:

$$\mathrm{P}^{(-h)}(Y \in \bullet) = \mathrm{P}^{(h)}(\theta_h(Y) \in \bullet) = \mathrm{P}^{(h)}((Y_{h+s}, s \in [-h,0]) \in \bullet). \tag{63}$$

We introduce the following assumptions.

($H6$) **There exists a probability measure on** $(D^-, \mathcal{D}_{(-\infty,0]})$ **denoted** $\mathrm{P}^{(-\infty)}$ **such that for all** $x \in E$**,** $t \geq 0$**, and** f **bounded and** $\mathcal{D}_{[-t,0]}$ **measurable:**

$$\mathrm{E}_x^{(-h)}\left[f(Y_{[-t,0]})\right] \xrightarrow[h\to+\infty]{} \mathrm{E}^{(-\infty)}\left[f(Y_{[-t,0]})\right].$$

($H7$) **For all** $t > 0$**, there exists a non negative function** g **such that for all** $x \in E$**, for all** $h > 0$**:**

$$v_h(x) - v_{h+t}(x) \leq g(h) \quad \text{and} \quad \int_1^\infty dr\; g(r) < \infty.$$

Note that the probability measure $\mathrm{P}^{(-\infty)}$ in ($H6$) does not depend on the starting point x.

We can now state the result on the convergence of the superprocess backward from the extinction time.

Theorem 5.9. *Under* ($H1$)-($H4$) *and* ($H6$).

(i) *The distribution of the triplet* $\left(\theta_h(W_{T_{max}})_{[-t,0]}, \theta_h(R_g^{T_{max}})_{[-t,0]}, \theta_h(R_d^{T_{max}})_{[-t,0]}\right)$ *under* $\mathbf{N}_x^{(h)}$ *converges weakly to the distribution of the triplet* $(Y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]})$ *where* Y *has distribution* $\mathrm{P}^{(-\infty)}$ *and conditionally on* Y, $R^{W,g}$ *and* $R^{W,d}$ *are two independent Poisson point measures with intensity:*

$$\mathbf{1}_{\{s<0\}}\alpha(Y_s)\; ds\; \mathbf{1}_{\{H_{max}(W)<-s\}}\; \mathbf{N}_{Y_s}[dW].$$

We even have the slightly stronger result. For any bounded measurable function F, *we have:*

$$\mathbf{N}_x^{(h)}\left[F\left(\theta_h(W_{T_{max}})_{[-t,0]}, \theta_h(R_g^{T_{max}})_{[-t,0]}, \theta_h(R_d^{T_{max}})_{[-t,0]}\right)\right] \xrightarrow[h\to+\infty]{} \mathrm{E}^{(-\infty)}\left[F\left(Y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]}\right)\right]. \tag{64}$$

(ii) *If furthermore* ($H7$) *holds, then the process* $\theta_h(X)_{[-t,0]} = (X_{h+s}, s \in [-t,0])$ *under* $\mathbb{N}_x^{(h)}$ *weakly converges towards* $X^{(-\infty)}_{[-t,0]}$, *where for* $s \leq 0$*:*

$$X_s^{(-\infty)} = \sum_{j\in J,\; s_j<s} X^j_{s-s_j},$$

and conditionally on Y *with distribution* $\mathrm{P}^{(-\infty)}$, $\sum_{j\in J} \delta_{(s_j,X^j)}$ *is a Poisson point measure with intensity:*

$$2\; \mathbf{1}_{\{s<0\}}\alpha(Y_s)\; ds\; \mathbf{1}_{\{H_{max}(X)<-s\}}\; \mathbb{N}_{Y_s}[dX].$$

Remark 5.10. We provide in Lemmas 7.3 and 7.6 sufficient conditions for ($H6$) and ($H7$) to hold in the case of the multitype Feller diffusion and the superdiffusion. These conditions are stated in term of the generalized eigenvalue λ_0 defined in (57) and its associated eigenfunction.

Proof. Let $0 < t < h$. We use notations from Theorems 4.12, 5.5 and 5.9. Let F be a bounded measurable function on $\mathcal{W}^- \times (\mathbb{R}^- \times \bar{\Omega})^2$ with $\mathcal{W}^-$ the set of killed paths indexed by negative times. We want to control δ_h defined by:

$$\delta_h = \mathbf{N}_x^{(h)}\Big[F\big(\theta_h(W_{T_{\max}})_{[-t,0]}, \theta_h(R^g_{T_{\max}})_{[-t,0]}, \theta_h(R^d_{T_{\max}})_{[-t,0]}\big)\Big] - \mathrm{E}^{(-\infty)}\Big[F\big(Y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]}\big)\Big].$$

We set:

$$\Upsilon(y_{[-t,0]}) = \mathrm{E}^{(-\infty)}\Big[F\big(y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]}\big)\Big| Y = y\Big].$$

We deduce from Williams' decomposition, Theorem 4.12, and the definition of $R^{W,g}$ and $R^{W,d}$, that:

$$\mathbf{N}_x^{(h)}\Big[F\big(\theta_h(W_{T_{\max}})_{[-t,0]}, \theta_h(R^g_{T_{\max}})_{[-t,0]}, \theta_h(R^d_{T_{\max}})_{[-t,0]}\big)\Big] = \mathrm{E}_x^{(-h)}\big[\Upsilon(Y_{[-t,0]})\big].$$

We thus can rewrite δ_h as:

$$\delta_h = \mathrm{E}_x^{(-h)}\big[\Upsilon(Y_{[-t,0]})\big] - \mathrm{E}^{(-\infty)}\big[\Upsilon(Y_{[-t,0]})\big].$$

The function Υ being bounded by $\|F\|_\infty$ and measurable, we may conclude under assumption $(H6)$ that $\lim_{h\to+\infty} \delta_h = 0$. This proves point (i).

We now prove point (ii). Let $t > 0$ and $\varepsilon > 0$ be fixed. Let F be a bounded measurable function on the space of continuous measure-valued applications indexed by negative times. For a point measure on $\mathbb{R}^- \times \bar{\Omega}$, $M = \sum_{i\in\mathcal{I}} \delta_{(s_i,W_i)}$, we set:

$$\tilde{F}(M) = F\left(\big(\sum_{i\in\mathcal{I}} \theta_{s_i}(X(W_i))\big)_{[-t,0]}\right).$$

For $h > t$, we want a control of $\bar{\delta}_h$ defined by:

$$\bar{\delta}_h = \mathbb{N}_x^{(h)}\Big[F\big(\theta_h(X)_{[-t,0]}\big)\Big] - \mathrm{E}^{(-\infty)}\Big[\tilde{F}\big(R^{W,g} + R^{W,d}\big)\Big].$$

By Corollary 4.13, we have:

$$\mathbb{N}_x^{(h)}\Big[F\big(\theta_h(X)_{[-t,0]}\big)\Big] = \mathbf{N}_x^{(h)}\Big[\tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})\big)\Big].$$

Thus, we get:

$$\bar{\delta}_h = \mathbf{N}_x^{(h)}\Big[\tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})\big)\Big] - \mathrm{E}^{(-\infty)}\Big[\tilde{F}\big(R^{W,g} + R^{W,d}\big)\Big]. \tag{65}$$

For $a > s$ fixed, we introduce $\bar{\delta}_h^a$, for $h > a$, defined by:

$$\bar{\delta}_h^a = \mathbf{N}_x^{(h)}\Big[\tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})_{[-a,0]}\big)\Big] - \mathrm{E}^{(-\infty)}\Big[\tilde{F}\big((R^{W,g} + R^{W,d})_{[-a,0]}\big)\Big]. \tag{66}$$

Notice the restriction of the point measures to $[-a, 0]$. Point (i) directly yields that $\lim_{h\to+\infty} \bar{\delta}_h^a = 0$. Thus, there exists $h_a > 0$ such that for all $h \geq h_a$,

$$\bar{\delta}_h^a \leq \varepsilon/2.$$

We now consider the difference $\bar{\delta}_h - \bar{\delta}_h^a$. We associate to the point measures M introduced above the most recent common ancestor of the population alive at time $-t$:

$$A(M) = \sup\{s > 0; \sum_{i\in\mathcal{I}} \mathbf{1}_{\{s_i<-s\}} \mathbf{1}_{\{H_{\max}(W_i)>-t-s_i\}} \neq 0\}.$$

Let us observe that:

$$\text{(67)} \qquad \mathbf{N}_x^{(h)} \text{ a.s.,} \quad \tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})\big)\mathbf{1}_{\{A\le a\}} = \tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})_{[-a,0]}\big)\mathbf{1}_{\{A\le a\}},$$

with $A = A(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})_{[-h,0]})$ in the left and in the right hand side. Similarly, we have:

$$\text{(68)} \qquad \mathrm{P}^{(-\infty)} \text{ a.s.,} \quad \tilde{F}\big(R^{W,g} + R^{W,d}\big)\mathbf{1}_{\{A\le a\}} = \tilde{F}\big((R^{W,g} + R^{W,d})_{[-a,0]}\big)\mathbf{1}_{\{A\le a\}},$$

with $A = A\big(R^{W,g} + R^{W,d}\big)$ in the left and in the right hand side. We thus deduce the following bound on $\bar{\delta}_h - \bar{\delta}_h^a$:

$$\begin{aligned}
|\bar{\delta}_h - \bar{\delta}_h^a| &\le 2\|F\|_\infty \Big[\mathbf{N}_x^{(h)}\big[A > a\big] + \mathrm{P}^{(-\infty)}\big[A > a\big]\Big] \\
&= 2\|F\|_\infty \Big[\mathrm{E}_x^{(-h)}\big[1 - \mathrm{e}^{-\int_a^h dr\ 2\alpha(v_{r-t}-v_r)(Y_{-r})}\big] + \mathrm{E}^{(-\infty)}\big[1 - \mathrm{e}^{-\int_a^\infty dr\ 2\alpha(v_{r-t}-v_r)(Y_{-r})}\big]\Big] \\
&\le 8\|F\|_\infty \|\alpha\|_\infty \int_{a-t}^\infty dr\ g(r),
\end{aligned}$$

where we used (65), (66), (67) and (68) for the first inequality, the definition of A for the first equality, as well as $(H7)$ and the fact that $1-\mathrm{e}^{-x} \le x$ if $x \ge 0$ for the last inequality. From $(H7)$, we can choose a large enough such that: $|\bar{\delta}_h - \bar{\delta}_h^a| \le \varepsilon/2$. We deduce that for all $h \ge \max(a, h_a)$: $|\bar{\delta}_h| \le |\bar{\delta}_h - \bar{\delta}_h^a| + |\bar{\delta}_h^a| \le \varepsilon$. This proves point (ii). □

6. The assumptions $(H4)$, $(H5)_\nu$ and $(H6)$

We assume in all this section that P **is the distribution of a diffusion in** $\mathbb{R}^K$ **for** K **integer or the law of a finite state space Markov Chain,** see Section 7 and the references therein. In particular, the generalized eigenvalue λ_0 of $(\beta - \mathcal{L})$ (see (86) or (88)) is known to exist. We will denote by ϕ_0 the associated right eigenvector. We shall consider the assumption:

$(H8)$ **There exist two positive constants** C_1 **and** C_2 **such that** $\forall x \in E,\ C_1 \le \phi_0(x) \le C_2$**; and** $\phi_0 \in \mathcal{D}(\mathcal{L})$**.**

Under $(H8)$, let $\mathrm{P}_x^{\phi_0}$ be the probability measure on $(D, \mathcal{D})$ defined by (9) with g replaced by ϕ_0:

$$\text{(69)} \qquad \forall t \ge 0, \quad \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\phi_0(Y_t)}{\phi_0(Y_0)}\, \mathrm{e}^{-\int_0^t ds\ (\beta(Y_s)-\lambda_0)}.$$

We shall also consider the assumption:

$(H9)$ **The probability measure** P^{ϕ_0} **admits a stationary measure** π**, and we have:**

$$\text{(70)} \qquad \sup_{f\in b\mathcal{E}, \|f\|_\infty\le 1} |\mathrm{E}_x^{\phi_0}[f(Y_t)] - \pi(f)| \xrightarrow[t\to+\infty]{} 0.$$

Notice the two hypotheses $(H8)$ and $(H9)$ hold for the examples of Section 7, see Lemmas 7.1 and 7.5.

Let us mention at this point that we will check that $\mathrm{P}_x^{\phi_0} = \mathrm{P}_x^{(\infty)}$ with $\mathrm{P}_x^{(\infty)}$ defined by (58), see Proposition 6.8.

6.1. **Proof of** $(H4)$**-**$(H6)$**.** Notice $(H9)$ implies that the probability measure $\mathrm{P}_\pi^{\phi_0}$ admits a stationary version on $D(\mathbb{R}, E)$, which we still denote by $\mathrm{P}_\pi^{\phi_0}$.

We introduce a specific h-transform of the superprocess. From Proposition 3.5 and the definition of the generalized eigenvalue (86) and (88), we have that the h-transform given by Definition 3.4 with $g = \phi_0$ of the $(\mathcal{L}, \beta, \alpha)$ superprocess is the $(\mathcal{L}^{\phi_0}, \lambda_0, \alpha\phi_0)$ superprocess. We define v^{ϕ_0} for all $t > 0$ and $x \in E$ by:

$$v_t^{\phi_0}(x) = \mathbb{N}_x^{(\mathcal{L}^{\phi_0},\lambda_0,\alpha\phi_0)}[H_{\max} > t]. \tag{71}$$

Observe that, as in (17), the following normalization holds between v^{ϕ_0} and v:

$$v_t^{\phi_0}(x) = \frac{v_t(x)}{\phi_0(x)}. \tag{72}$$

Our first task is to give precise bounds on the decay of $v_t^{\phi_0}$ as t goes to ∞.

We first offer bounds for the case $\lambda_0 = 0$ in Lemma 6.1, relying on a coupling argument. This in turn gives sufficient condition under which $(H1)$ holds in Lemma 6.2 We then give Feynman-Kac representation formulae, Lemma 6.3, which yield exponential bounds in the case $\lambda_0 > 0$, see Lemma 6.4. We finally strengthen in Lemma 6.6 the bound of Lemma 6.4 by proving the exponential behavior of $v_t^{\phi_0}$ in the case $\lambda_0 > 0$. The proofs of Lemmas 6.1, 6.2, 6.3, 6.4 and 6.6 are given in Section 6.2.

We first give a bound in the case $\lambda_0 = 0$.

Lemma 6.1. *Assume* $\lambda_0 = 0$, $(H2)$ *and* $(H8)$. *Then for all* $t > 0$*:*

$$\alpha\phi_0(x)\frac{1}{\|\alpha\phi_0\|_\infty^2} \leq t\, v_t^{\phi_0}(x) \leq \alpha\phi_0(x) \left\| \frac{1}{\alpha\phi_0} \right\|_\infty^2 .$$

A coupling argument then implies that $(H1)$ holds:

Lemma 6.2. *Assume* $\lambda_0 \geq 0$, $(H2)$ *and* $(H8)$. *Then* $(H1)$ *holds.*

We give a Feynman-Kac's formula for v_0^ϕ and ∂v_0^ϕ.

Lemma 6.3. *Assume* $\lambda_0 \geq 0$, $(H2)$-$(H3)$ *and* $(H8)$. *Let* $\varepsilon > 0$. *We have:*

$$v_{h+\varepsilon}^{\phi_0}(x) = \mathrm{e}^{-\lambda_0 h}\, \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\, v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\, v_\varepsilon^{\phi_0}(Y_h)\right], \tag{73}$$

$$\partial_h v_{h+\varepsilon}^{\phi_0}(x) = \mathrm{e}^{-\lambda_0 h}\, \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-2\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\, v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\, \partial_h v_\varepsilon^{\phi_0}(Y_h)\right]. \tag{74}$$

We give exponential bounds for v_0^ϕ and $\partial_t v_0^\phi$ in the subcritical case.

Lemma 6.4. *Assume* $\lambda_0 > 0$, $(H2)$-$(H3)$ *and* $(H8)$. *Fix* $t_0 > 0$. *There exists* C_3 *and* C_4 *two positive constants such that, for all* $x \in E$, $t > t_0$*:*

$$C_3 \leq v_t^{\phi_0}(x)\, \mathrm{e}^{\lambda_0 t} \leq C_4. \tag{75}$$

There exists C_5 *and* C_6 *two positive constants such that, for all* $x \in E$, $t > t_0$*:*

$$C_5 \leq |\partial_t v_t^{\phi_0}(x)|\, \mathrm{e}^{\lambda_0 t} \leq C_6. \tag{76}$$

As a direct consequence of (75), we get the following Lemma.

Lemma 6.5. *Assume* $\lambda_0 > 0$, $(H2)$-$(H3)$ *and* $(H8)$. *Then* $(H7)$ *holds.*

In what follows, the notation $o_h(1)$ refers to any function F_h such that $\lim_{h\to+\infty} \| F_h \|_\infty = 0$. We now improve on Lemma 6.4, by using the ergodic formula (70).

Lemma 6.6. *Assume $\lambda_0 \geq 0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$ hold. Then for all $\varepsilon > 0$, we have:*

$$\partial_t v^{\phi_0}_{h+\varepsilon}(x)\ \mathrm{e}^{\lambda_0 h} = \mathrm{E}^{\phi_0}_\pi \left[\mathrm{e}^{-2\int_0^h ds\ \alpha\, \phi_0\, v^{\phi_0}_{s+\varepsilon}(Y_{-s})}\ \partial_t v^{\phi_0}_\varepsilon(Y_0)\right] (1 + o_h(1)). \tag{77}$$

In addition, for $\lambda_0 > 0$, we have that:

$$\mathrm{E}^{\phi_0}_\pi \left[\mathrm{e}^{-2\int_0^\infty ds\ \alpha\, \phi_0\, v^{\phi_0}_{s+\varepsilon}(Y_{-s})}\ \partial_t v^{\phi_0}_\varepsilon(Y_0)\right]$$

is finite (notice the integration is up to $+\infty$) and:

$$\partial_t v^{\phi_0}_{h+\varepsilon}(x)\ \mathrm{e}^{\lambda_0 h} = \mathrm{E}^{\phi_0}_\pi \left[\mathrm{e}^{-2\int_0^\infty ds\ \alpha\, \phi_0\, v^{\phi_0}_{s+\varepsilon}(Y_{-s})}\ \partial_t v^{\phi_0}_\varepsilon(Y_0)\right] + o_h(1). \tag{78}$$

Our next goal is to prove $(H4)$ from $(H8)$-$(H9)$, see Proposition 6.8.

Fix $x \in E$. We observe from (54) and (69) that $\mathrm{P}^{(h)}_x$ is absolutely continuous with respect to $\mathrm{P}^{\phi_0}_x$ on $\mathcal{D}_{[0,t]}$ for $0 \leq t < h$. We define $M^{(h),\phi_0}_t$ the corresponding Radon-Nikodym derivative:

$$M^{(h),\phi_0}_t = \frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,t]}}}.$$

Using (54), (69) and the normalization $v(x) = v^{\phi_0}(x)\ \phi_0(x)$, we get:

$$\begin{aligned} M^{(h),\phi_0}_t &= \frac{\partial_t v_{h-t}(Y_t)\, \mathrm{e}^{-\lambda_0 t}}{\partial_t v_h(Y_0)} \frac{\phi_0(Y_0)}{\phi_0(Y_t)}\ \mathrm{e}^{-2\int_0^t ds\ \alpha(Y_s)\ v_{h-s}(Y_s)} \\ &= \frac{\partial_t v^{\phi_0}_{h-t}(Y_t)\, \mathrm{e}^{-\lambda_0 t}}{\partial_t v^{\phi_0}_h(Y_0)}\ \mathrm{e}^{-2\int_0^t ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{h-s}(Y_s)}\ . \end{aligned} \tag{79}$$

We have the following result on the convergence of $M^{(h),\phi_0}_t$.

Lemma 6.7. *Assume $(H2)$-$(H3)$ and $(H8)$-$(H9)$. For $\lambda_0 \geq 0$, we have:*

$$M^{(h),\phi_0}_t \xrightarrow[h\to+\infty]{} 1 \quad \mathrm{P}^{\phi_0}_x\text{-a.s. and in } \mathrm{L}^1(\mathrm{P}^{\phi_0}_x),$$

and for $\lambda_0 > 0$, we have:

$$M^{(h),\phi_0}_{h/2} \xrightarrow[h\to+\infty]{} 1 \quad \mathrm{P}^{\phi_0}_x\text{-a.s. and in } \mathrm{L}^1(\mathrm{P}^{\phi_0}_x).$$

Proof. We compute:

$$\begin{aligned} M^{(h),\phi_0}_t &= \frac{\partial_t v^{\phi_0}_{h-t}(Y_t)\, \mathrm{e}^{\lambda_0(h-t)}}{\partial_t v^{\phi_0}_h(Y_0)\, \mathrm{e}^{\lambda_0 h}}\ \mathrm{e}^{-2\int_0^t ds\ \alpha(Y_s)\phi_0(Y_s) v^{\phi_0}_{h-s}(Y_s)} \\ &= \frac{\mathrm{E}^{\phi_0}_\pi \left[\mathrm{e}^{-2\int_{-(h-t-\varepsilon)}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\ \partial_h v^{\phi_0}_\varepsilon(Y_0)\right] \big(1+o_h(1)\big)}{\mathrm{E}^{\phi_0}_\pi \left[\mathrm{e}^{-2\int_{-h-\varepsilon}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\ \partial_h v^{\phi_0}_\varepsilon(Y_0)\right] \big(1+o_h(1)\big)} \big(1+o_h(1)\big) \\ &= \frac{\mathrm{E}^{\phi_0}_\pi \left[\mathrm{e}^{-2\int_{-h-\varepsilon}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\ \partial_h v^{\phi_0}_\varepsilon(Y_0)\right]}{\mathrm{E}^{\phi_0}_\pi \left[\mathrm{e}^{-2\int_{-h-\varepsilon}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\ \partial_h v^{\phi_0}_\varepsilon(Y_0)\right]} \big(1+o_h(1)\big) \\ &= 1 + o_h(1), \end{aligned}$$

where we used (79) for the first equality, (77) twice and the boundedness of α and ϕ_0 as well as the convergence of v_h to 0 for the second, and Lemma 6.4 (if $\lambda_0 > 0$) or Lemma 6.1 (if $\lambda_0 = 0$)

for the fourth. Since $o_h(1)$ is bounded and converges uniformly to 0, we get that the convergence of $M_t^{(h),\phi_0}$ towards 1 holds $\mathrm{P}_x^{\phi_0}$-a.s. and in $L^1(\mathrm{P}_x^{\phi_0})$.

Similar arguments relying on (78) instead of (77) imply that $M_{h/2}^{(h),\phi_0} = 1 + o_h(1)$ for $\lambda_0 > 0$. Since $o_h(1)$ is bounded and converges uniformly to 0, we get that the convergence of $M_{h/2}^{(h),\phi_0}$ towards 1 holds $\mathrm{P}_x^{\phi_0}$-a.s. and in $L^1(\mathrm{P}_x^{\phi_0})$. □

The previous Lemma enables us to conclude about $(H4)$.

Proposition 6.8. *Assume $\lambda_0 \geq 0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$. Then $(H4)$ holds with $\mathrm{P}_x^{(\infty)} = \mathrm{P}_x^{\phi_0}$.*

Proof. Notice that:

$$M_t^{(h)} = \frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}_{x\ |\mathcal{D}_{[0,t]}}} = M_t^{(h),\phi_0} \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}_{x\ |\mathcal{D}_{[0,t]}}}.$$

The convergence $\lim_{h\to+\infty} M_t^{(h),\phi_0} = 1$ $\mathrm{P}_x^{\phi_0}$-a.s. and in $\mathrm{L}^1(\mathrm{P}_x^{\phi_0})$ readily implies $(H4)$. Then, use (58) to get $\mathrm{P}^{(\infty)} = \mathrm{P}^{\phi_0}$. □

Notice that $(H5)_\nu$ is a direct consequence of Lemma 6.6.

Corollary 6.9. *Assume $\lambda_0 \geq 0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$. Then $(H5)_\nu$ holds with $\rho = \phi_0/\nu(\phi_0)$.*

Proof. We deduce from (72) and (77) that:

$$\partial_t v_h(x) = f(h)\phi_0(x)\ (1 + o_h(1))\, \mathrm{e}^{-\lambda_0 h},$$

for some positive function f of h. Then we get:

$$\frac{\partial_h v_h(x)}{\nu(\partial_h v_h)} = \frac{\phi_0(x)}{\nu(\phi_0)}(1 + o_h(1)).$$

This gives $(H5)_\nu$, as $o_h(1)$ is bounded, with $\rho = \phi_0/\nu(\phi_0)$. □

Our next goal is to prove $(H6)$ from $(H8)$-$(H9)$, see Proposition 6.12.

Observe from (53), (63) and (69) that $\mathrm{P}_\pi^{(-h)}$ is absolutely continuous with respect to $\mathrm{P}_\pi^{\phi_0}$ on $\mathcal{D}_{[-h,0]}$. We define $L^{(-h)}$ the corresponding Radon-Nikodym derivative:

$$L^{(-h)} = \frac{d\mathrm{P}^{(-h)}_{\pi\ |\mathcal{D}_{[-h,0]}}}{d\mathrm{P}^{\phi_0}_{\pi\ |\mathcal{D}_{[-h,0]}}} = \frac{1}{\alpha(Y_0)\phi_0(Y_0)}\, \frac{\partial_h v_h^0\, \mathrm{e}^{\beta_0 h}}{\partial_h v_h^{\phi_0}(Y_{-h})\, \mathrm{e}^{\lambda_0 h}}\, \mathrm{e}^{-2\int_{-h}^0 (\alpha(Y_s) v_{-s}(Y_s) - v_{-s}^0)\, ds}. \tag{80}$$

The next Lemma insures the convergence of $L^{(-h)}$ to a limit, say $L^{(-\infty)}$.

Lemma 6.10. *Assume $\lambda_0 > 0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$. We have:*

$$L^{(-h)} \xrightarrow[h\to+\infty]{} L^{(-\infty)} \quad \mathrm{P}_\pi^{\phi_0}\text{-a.s. and in } \mathrm{L}^1(\mathrm{P}_\pi^{\phi_0}).$$

Proof. Notice that $\lim_{h\to+\infty} \partial_h v_h^0\, \mathrm{e}^{\beta_0 h} = -\beta_0^2$. We also deduce from (48), (49) and (75) that $\int_{-h}^0 (\alpha(Y_s) v_{-s}(Y_s) - v_{-s}^0)\, ds$ increases, as h goes to infinity; to $\int_{-\infty}^0 (\alpha(Y_s) v_{-s}(Y_s) - v_{-s}^0)\, ds$ which is finite. For fixed $t > 0$, we also deduce from (78) (with h replaced by $h - t$ and ε by t) that $\mathrm{P}_\pi^{\phi_0}$ a.s.:

$$\lim_{h\to+\infty} \partial_t v_h^{\phi_0}(Y_{-h})\, \mathrm{e}^{\lambda_0 h} = \mathrm{e}^{\lambda_0 t}\, \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-\infty}^{-t} ds\ \alpha(Y_s)\ \phi_0(Y_s) v_{-s}^{\phi_0}(Y_s)}\, \partial_t v_t^{\phi_0}(Y_{-t})\right].$$

We deduce from (80) the $\mathrm{P}_\pi^{\phi_0}$ a.s. convergence of $(L^{(-h)}, h > 0)$ to $L^{(-\infty)}$. Notice from (76) that, for fixed t, the sequence $(L^{(-h)}, h > t)$ is bounded. Hence the previous convergence holds also in $\mathrm{L}^1(\mathrm{P}_\pi^{\phi_0})$. □

As $\mathrm{E}_\pi^{\phi_0}\left[L^{(-h)}\right] = 1$, we deduce that $\mathrm{E}_\pi^{\phi_0}\left[L^{(-\infty)}\right] = 1$. We define the probability measure $\mathrm{P}_\pi^{(-\infty),\phi_0}$ on $(D^-, \mathcal{D}_{(-\infty,0]})$ by its Radon Nikodym derivative:

$$\frac{d\mathrm{P}^{(-\infty),\phi_0}_{\pi\ |\mathcal{D}_{(-\infty,0]}}}{d\mathrm{P}^{\phi_0}_{\pi\ |\mathcal{D}_{(-\infty,0]}}} = L^{(-\infty)}. \tag{81}$$

Remark 6.11. Assume $\lambda_0 > 0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$. Define for $h > t > 0$:

$$L_{-t}^{(-h)} = \mathrm{E}_\pi^{\phi_0}[L^{(-h)}|\mathcal{D}_{(-\infty,-t]}] = \frac{d\mathrm{P}^{(-h)}_{\pi\ |\mathcal{D}_{[-h,-t]}}}{d\mathrm{P}^{\phi_0}_{\pi\ |\mathcal{D}_{[-h,-t]}}}$$

$$L_{-t}^{(-\infty)} = \mathrm{E}_\pi^{\phi_0}[L^{(-\infty)}|\mathcal{D}_{(-\infty,-t]}] = \frac{d\mathrm{P}^{(-\infty),\phi_0}_{\pi\ |\mathcal{D}_{(-\infty,-t]}}}{d\mathrm{P}^{\phi_0}_{\pi\ |\mathcal{D}_{(-\infty,-t]}}}.$$

Using (55) and Lemma 6.3, we get:

$$\begin{aligned} L_{-t}^{(-h)} &= \frac{\partial_t v_t(Y_{-t})}{\partial_t v_h(Y_{-h})} \frac{\phi_0(Y_{-h})}{\phi_0(Y_{-t})} \mathrm{e}^{-\lambda_0(h-t)} \mathrm{e}^{-2\int_{-h}^{-t} ds\ \alpha(Y_s)\, v_{-s}(Y_s)} \\ &= \frac{\mathrm{e}^{-2\int_{-h}^{-t} ds\ \alpha(Y_s)\ \phi_0(Y_s)\, v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})}{\mathrm{E}_{Y_{-h}}^{\phi_0}\left[\mathrm{e}^{-2\int_{-h}^{-t} ds\ \alpha(Y_s)\ \phi_0(Y_s)\, v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})\right]}. \end{aligned}$$

Using Lemma 6.6 and convergence of $(L_{-t}^{(-h)}, h > t)$ to $L_{-t}^{(-\infty)}$, which is a consequence of Lemma 6.10, we also get that for $t > 0$:

$$L_{-t}^{(-\infty)} = \frac{\mathrm{e}^{-2\int_{-\infty}^{-t} ds\ \alpha(Y_s)\ \phi_0(Y_s)\, v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})}{\mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-\infty}^{-t} ds\ \alpha(Y_s)\ \phi_0(Y_s)\, v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})\right]}.$$

Those formulas are more self-contained than (80) and the definition of $L^{(-\infty)}$ as a limit, but they only hold for $t > 0$.

The following Proposition gives that $(H6)$ holds.

Proposition 6.12. *Assume $\lambda_0 > 0$, $(H2)$-$(H3)$ and $(H8)$-$(H9)$. Then $(H6)$ holds with $\mathrm{P}^{(-\infty)} = \mathrm{P}^{(-\infty),\phi_0}$.*

Proof. Let $0 < t$ and F be a bounded and $\mathcal{D}_{[-t,0]}$ measurable function. For h large enough, we have:

$$\begin{aligned} \mathrm{E}_x^{(-h)}\left[F(Y_{[-t,0]})\right] &= \mathrm{E}_x^{(h)}\left[\mathrm{E}_{Y_{h/2}}^{(h/2)}\left[F(\theta_{h/2}(Y)_{[-t,-s]})\right]\right] \\ &= \mathrm{E}_x^{\phi_0}\left[M_{h/2}^{(h),\phi_0}\ \mathrm{E}_{Y_{h/2}}^{(h/2)}\left[F(\theta_{h/2}(Y)_{[-t,0]})\right]\right] \\ &= \mathrm{E}_x^{\phi_0}\left[\mathrm{E}_{Y_{h/2}}^{(h/2)}\left[F(\theta_{h/2}(Y)_{[-t,0]})\right]\right] + o_h(1) \\ &= \mathrm{E}_\pi^{(h/2)}\left[F(\theta_{h/2}(Y)_{[-t,0]})\right] + o_h(1) \\ &= \mathrm{E}_\pi^{(-h/2)}\left[F(Y_{[-t,0]})\right] + o_h(1), \end{aligned}$$

where we used the definition of $\mathrm{P}^{(-h)}$ and the Markov property for the first equality, Lemma 6.7 together with F bounded by $\|F\|_\infty$ for the third, and assumption $(H9)$ for the fourth. We continue the computations as follows:

$$\begin{aligned}\mathrm{E}_x^{(-h)}\left[F(Y_{[-t,0]})\right] &= \mathrm{E}_\pi^{\phi_0}\left[L^{(-h/2)}F(Y_{[-t,0]})\right] + o_h(1)\\ &= \mathrm{E}_\pi^{\phi_0}\left[L^{(-\infty)}F(Y_{[-t,0]})\right] + o_h(1)\\ &= \mathrm{E}_\pi^{(-\infty),\phi_0}\left[F(Y_{[-t,0]})\right] + o_h(1),\end{aligned}$$

where we used Lemma 6.10 for the second equality. This gives $(H6)$ with $\mathrm{P}^{(-\infty)} = \mathrm{P}^{(-\infty),\phi_0}$. $\square$

6.2. **Proof of Lemmas 6.1, 6.2, 6.3, 6.4 and 6.6.**

Proof of Lemma 6.1. From $(H2)$ and $(H8)$, there exist $m, M \in \mathbb{R}$ such that

$$\forall x \in E, 0 < m \le \alpha\phi_0(x) \le M < \infty.$$

Let W be a $(\frac{M}{\alpha\phi_0}\mathcal{L}, 0, M)$ Brownian snake and define the time change Φ for every $w \in \mathcal{W}$ by $\Phi_t(w) = \int_0^t ds\ \frac{M}{\alpha\phi_0}(w(s))$. As $\partial_t\Phi_t(w) \ge 1$, we have that $t \to \Phi_t(w)$ is strictly increasing. Let $t \to \Phi_t^{(-1)}(w)$ denote its inverse. Then, using Proposition 12 of [10], first step of the proof, we have that the time changed snake $W \circ \Phi^{-1}$, with value

$$(W \circ \Phi^{-1})_s = (W_s(\Phi_t^{-1}(W_s)), t \in [0, \Phi^{-1}(W_s, H_s)])$$

at time s, is a $(\mathcal{L}, 0, \alpha\phi_0)$ Brownian snake. Noting the obvious bound on the time change $\Phi_t^{-1}(w) \le t$, we have, according to Theorem 14 of [10]:

$$\mathbf{P}^{\left(\frac{M}{\alpha\phi_0}\ \mathcal{L}^{\phi_0},0,M\right)}_{\frac{\alpha\phi_0(x)}{M}\delta_x}(H_{\max} \le t) \ge \mathbf{P}^{(\mathcal{L}^{\phi_0},0,\alpha\phi_0)}_{\delta_x}(H_{\max} \le t)$$

which implies:

$$\frac{\alpha\phi_0(x)}{M}\,\mathbf{N}_x^{\left(\frac{M}{\alpha\phi_0}\mathcal{L}^{\phi_0},0,M\right)}(H_{\max} > t) \le \mathbf{N}_x^{(\mathcal{L}^{\phi_0},0,\alpha\phi_0)}(H_{\max} > t)$$

from the exponential formula for Poisson point measures. Now, the left hand side of this inequality can be computed explicitly:

$$\mathbf{N}_x^{\left(\frac{M}{\alpha\phi_0}\mathcal{L}^{\phi_0},0,M\right)}(H_{\max} > t) = \mathbb{N}_x^{\left(\frac{M}{\alpha\phi_0}\mathcal{L}^{\phi_0},0,M\right)}(H_{\max} > t) = \frac{1}{Mt}$$

and the right hand side of this inequality is $v_t^{\phi_0}(x)$ from (71). We thus have proved that:

$$\frac{\alpha\phi_0(x)}{M^2 t} \le v_t^{\phi_0}(x),$$

and this yields the first part of the inequality of Lemma 6.1. The second part is obtained in the same way using the coupling with the $\left(\frac{m}{\alpha\phi_0}\mathcal{L}^{\phi_0}, 0, m\right)$ Brownian snake. $\square$

Proof of lemma 6.2. Assumption $(H2)$ and $(H8)$ allow us to apply Lemma 6.1 for the case $\lambda_0 = 0$, which yields that $v_\infty^{\phi_0} = 0$, and then $v_\infty = 0$ thanks to (72). This in turn implies that $(H1)$ holds in the case $\lambda_0 = 0$ according to Lemma 2.5. For $\lambda_0 > 0$, we may use item 5 of Proposition 13 of [10] (which itself relies on a Girsanov theorem) with $\mathbb{P}^{(\mathcal{L},0,\alpha\phi_0)}$ in the rôle of $\mathbb{P}^c$ and $\mathbb{P}^{(\mathcal{L}^{\phi_0},\lambda_0,\alpha\phi_0)}$ in the rôle of $\mathbb{P}^{b,c}$ to conclude that the extinction property $(H1)$ holds under $\mathbb{P}^{(\mathcal{L}^{\phi_0},\lambda_0,\alpha\phi_0)}$. $\square$

Proof of Lemma 6.3. Let $\varepsilon > 0$. The function v^{ϕ_0} is known to solve the following mild form of the Laplace equation, see equation (3):

$$v^{\phi_0}_{t+s}(x) + \mathrm{E}^{\phi_0}_x\left[\int_0^t dr\ \left(\lambda_0\, v^{\phi_0}_{t+s-r}(Y_r) + \alpha(Y_r)\phi_0(Y_r)(v^{\phi_0}_{t+s-r}(Y_r))^2\right)\right] = \mathrm{E}^{\phi_0}_x\left[v^{\phi_0}_s(Y_t)\right].$$

By differentiating with respect to s and taking $t = t-s$, we deduce from dominated convergence and the bounds (46), (47) and (49) on $v^{\phi_0} = v/\phi_0$ and its time derivative (valid under the assumptions $(H1)$-$(H3)$) the following mild form on the time derivative $\partial_t v^{\phi_0}$:

$$\partial_t v^{\phi_0}_t(x) + \mathrm{E}^{\phi_0}_x\left[\int_0^{t-s} dr\ \left(\lambda_0\ + 2\alpha(Y_r)\phi_0(Y_r)v^{\phi_0}_{t-r}(Y_r)\right)\partial_t v^{\phi_0}_{t-r}(Y_r)\right] = \mathrm{E}^{\phi_0}_x\left[\partial_t v^{\phi_0}_s(Y_{t-s})\right].$$

From the Markov property, for fixed $t > 0$, the two following processes:

$$\left(v^{\phi_0}_{t-s}(Y_s) - \int_0^s dr\ \left(\lambda_0 + \alpha(Y_r)\phi_0(Y_r)v^{\phi_0}_{t-r}(Y_r)\right)\, v^{\phi_0}_{t-r}(Y_r), 0 \le s < t\right)$$

and

$$\left(\partial_t v^{\phi_0}_{t-s}(Y_s) - \int_0^s dr\ \left(\lambda_0\ + 2\alpha(Y_r)\phi_0(Y_r)v^{\phi_0}_{t-r}(Y_r)\right)\partial_t v^{\phi_0}_{t-r}(Y_r), 0 \le s < t\right)$$

are $\mathcal{D}_s$-martingale under $\mathrm{P}^{\phi_0}_\pi$. A Feynman-Kac manipulation, as done in the proof of Lemma 3.1, enables us to conclude that for fixed $t > 0$:

$$\left(v^{\phi_0}_{t-s}(Y_s)\,\mathrm{e}^{-\int_0^s dr\ \left(\lambda_0+\alpha(Y_r)\phi_0(Y_r)v^{\phi_0}_{t-r}(Y_r)\right)}, 0 \le s < t\right)$$

and

$$\left(\partial_t v^{\phi_0}_{t-s}(Y_s)\,\mathrm{e}^{-\int_0^s dr\ \left(\lambda_0\ +2\alpha(Y_r)\phi_0(Y_r)v^{\phi_0}_{t-r}(Y_r)\right)}, 0 \le s < t\right)$$

are $\mathcal{D}_s$-martingale under $\mathrm{P}^{\phi_0}_\pi$. Taking expectations at time $s = 0$ and $s = h$ with $t = h + \varepsilon$, we get the representations formulae stated in the Lemma:

$$v^{\phi_0}_{h+\varepsilon}(x) = \mathrm{e}^{-\lambda_0 h}\,\mathrm{E}^{\phi_0}_x\left[\mathrm{e}^{-\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v^{\phi_0}_{h+\varepsilon-s}(Y_s)}\ v^{\phi_0}_\varepsilon(Y_h)\right],$$

$$\partial_h v^{\phi_0}_{h+\varepsilon}(x) = \mathrm{e}^{-\lambda_0 h}\,\mathrm{E}^{\phi_0}_x\left[\mathrm{e}^{-2\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v^{\phi_0}_{h+\varepsilon-s}(Y_s)}\ \partial_h v^{\phi_0}_\varepsilon(Y_h)\right].$$

□

Proof of Lemma 6.4. Since $v^{\phi_0}_\varepsilon = v_\varepsilon/\phi_0 = \tilde{v}_\varepsilon/(\alpha\phi_0)$, we can conclude from (46), $(H2)$ and $(H8)$ that $v^{\phi_0}_\varepsilon$ is bounded from above and from below by positive constants. Similarly, we also get from (47), (48) and (49) that $|\partial_h \tilde{v}_\varepsilon|$ is bounded from above and from below by two positive constants. Thus, we have the existence of four positive constants, D_1, D_2, D_3 and D_4, such that, for all $x \in E$:

$$D_1 \le v^{\phi_0}_\varepsilon(x) \le D_2, \tag{82}$$

$$D_3 \le |\partial_t v^{\phi_0}_\varepsilon(x)| \le D_4. \tag{83}$$

From equations (73), (82) and the positivity of v^{ϕ_0}, we deduce that:

$$v^{\phi_0}_{h+\varepsilon}(x) \le D_2\,\mathrm{e}^{-\lambda_0 h}\,. \tag{84}$$

Putting back (84) into (73), we have the converse inequality $D_5\,\mathrm{e}^{-\lambda_0 h} \le v^{\phi_0}_{h+\varepsilon}(x)$ with $D_5 = D_1 \exp\{-D_2 \,\|\,\alpha\,\|_\infty\,\|\,\phi_0\,\|_\infty / \lambda_0\} > 0$. This gives (75).

Similar arguments using (74) and (83) instead of (73) and (82), gives (76). □

Proof of Lemma 6.6. Using the Feynman-Kac representation of $\partial_h v^{\phi_0}_{h+\varepsilon}$ from (73) as well as the Markov property, we have:

$$\begin{aligned}
\partial_h v^{\phi_0}_{h+\varepsilon}(x)\,\mathrm{e}^{\lambda_0 h} &= \mathrm{E}^{\phi_0}_x\left[\mathrm{e}^{-2\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v^{\phi_0}_{h+\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_h)\right]\\
&= \mathrm{E}^{\phi_0}_x\left[\mathrm{e}^{-2\int_0^{\sqrt{h}} ds\,\alpha\,\phi_0\,v^{\phi_0}_{h+\varepsilon-s}(Y_s)}\,\mathrm{E}^{\phi_0}_{Y_{\sqrt{h}}}\Big[\mathrm{e}^{-2\int_0^{h-\sqrt{h}} ds\,\alpha\,\phi_0\,v^{\phi_0}_{h-\sqrt{h}+\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_{h-\sqrt{h}})\Big]\right].
\end{aligned}$$

Notice that

$$\left|\int_0^{\sqrt{h}} ds\ \alpha\,\phi_0\,v^{\phi_0}_{h+\varepsilon-s}(Y_s)\right| \leq \|\alpha\|_\infty\,\|\phi_0\|_\infty\,\sqrt{h}\,\|v^{\phi_0}_{h+\varepsilon-\sqrt{h}}\|_\infty = o_h(1), \tag{85}$$

according to Lemma 6.4 if $\lambda_0 > 0$ and Lemma 6.1 if $\lambda_0 = 0$. We get:

$$\begin{aligned}
\partial_h v^{\phi_0}_{h+\varepsilon}(x)\,\mathrm{e}^{\lambda_0 h} &= \mathrm{E}^{\phi_0}_x\left[\mathrm{E}^{\phi_0}_{Y_{\sqrt{h}}}\Big[\mathrm{e}^{-2\int_0^{h-\sqrt{h}} ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v^{\phi_0}_{h-\sqrt{h}+\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_{h-\sqrt{h}})\Big]\right]\big(1+o_h(1)\big)\\
&= \mathrm{E}^{\phi_0}_\pi\left[\mathrm{e}^{-2\int_0^{h-\sqrt{h}} ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v^{\phi_0}_{h-\sqrt{h}+\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_{h-\sqrt{h}})\right]\big(1+o_h(1)\big)\\
&= \mathrm{E}^{\phi_0}_\pi\left[\mathrm{e}^{-2\int_{-(h-\sqrt{h})}^0 ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v^{\phi_0}_{\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_0)\right]\big(1+o_h(1)\big)\\
&= \mathrm{E}^{\phi_0}_\pi\left[\mathrm{e}^{-2\int_{-h}^0 ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v^{\phi_0}_{\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_0)\right]\big(1+o_h(1)\big),
\end{aligned}$$

where we used (85) for the first equality, $(H9)$ for the second, stationarity of Y under $\mathrm{P}^{\phi_0}_\pi$ for the third and (85) again for the last. This gives (77).

Moreover, if $\lambda_0 > 0$, we get that:

$$\mathrm{E}^{\phi_0}_\pi\left[\mathrm{e}^{-2\int_{-\infty}^0 ds\ \alpha\,\phi_0\,v^{\phi_0}_{\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_0)\right]$$

is finite and that:

$$\lim_{h'\to+\infty}\mathrm{E}^{\phi_0}_\pi\left[\mathrm{e}^{-2\int_{-h'}^0 ds\ \alpha\,\phi_0\,v^{\phi_0}_{\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_0)\right] = \mathrm{E}^{\phi_0}_\pi\left[\mathrm{e}^{-2\int_{-\infty}^0 ds\ \alpha\,\phi_0\,v^{\phi_0}_{\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_0)\right].$$

Therefore, we deduce (78) from (77). □

6.3. **About the Bismut spine.** Choosing uniformly an individual at random at height t under $\mathbf{N}_x$ and letting $t \to \infty$, we will see that the law of the ancestral lineage should converge in some sense to the law of the oldest ancestral lineage which itself converges to $\mathrm{P}^{(\infty)}_x$ defined in (58), according to Lemma 6.8.

We have defined in (45) the following family of probability measure indexed by $t \geq 0$:

$$\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\right]}.$$

Lemma 6.13. *Assume* $(H8)$-$(H9)$. *We have, for every* $0 \leq t_0 \leq t$:

$$\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}} \xrightarrow[t\to+\infty]{} \frac{d\mathrm{P}^{(\infty)}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}} \quad \mathrm{P}_x\text{-}a.s.\ \textit{and in}\ \mathrm{L}^1(\mathrm{P}_x).$$

Note that there is no restriction on the sign of λ_0 for this Lemma to hold.

Remark 6.14. This result correspond to the so called globular phase in the random polymers literature (see [8], Theorem 8.3).

Proof. We have:

$$\begin{aligned}
\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}} &= \mathrm{e}^{-\int_0^{t_0} ds\ \beta(Y_s)}\ \frac{\mathrm{E}_{Y_{t_0}}\left[\mathrm{e}^{-\int_0^{t-t_0} ds\ \beta(Y_s)}\right]}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^{t} ds\ \beta(Y_s)}\right]}\\
&= \mathrm{e}^{-\int_0^{t_0} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\mathrm{E}_{Y_{t_0}}\left[\mathrm{e}^{-\int_0^{t-t_0} ds\ (\beta(Y_s)-\lambda_0)}\right]}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^{t} ds\ (\beta(Y_s)-\lambda_0)}\right]}\\
&= \mathrm{e}^{-\int_0^{t_0} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\phi_0(Y_{t_0})}{\phi_0(Y_0)}\frac{\mathrm{E}_{Y_{t_0}}\left[\mathrm{e}^{-\int_0^{t-t_0} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\phi_0(Y_{t-t_0})}{\phi_0(Y_0)}\frac{1}{\phi_0(Y_{t-t_0})}\right]}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^{t} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\phi_0(Y_t)}{\phi_0(x)}\frac{1}{\phi_0(Y_t)}\right]}\\
&= \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}}\frac{\mathrm{E}^{\phi_0}_{Y_{t_0}}\left[1/\phi_0(Y_{t-t_0})\right]}{\mathrm{E}^{\phi_0}_x\left[1/\phi_0(Y_t)\right]}\\
&\underset{t\to\infty}{\longrightarrow} \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}}\frac{\pi(\frac{1}{\phi_0})}{\pi(\frac{1}{\phi_0})} = \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}},
\end{aligned}$$

where we use the Markov property at the first equality, we force the apparition of λ_0 at the second equality and we force the apparition of ϕ_0 at the third equality in order to obtain the Radon Nikodym derivative of $\mathrm{P}^{\phi_0}_x$ with respect to P_x: this observation gives the fourth equality. The ergodic assumption $(H9)$ ensures the P_x-a.s. convergence to 1 of the fraction in the fourth equality as t goes to ∞. Since

$$\left((t,y)\to \mathrm{E}^{\phi_0}_y\left[1/\phi_0(Y_{t-t_0})\right]/\mathrm{E}^{\phi_0}_x\left[1/\phi_0(Y_t)\right]\right)$$

is bounded according to $(H8)$, we conclude that the convergence also holds in $\mathrm{L}^1(\mathrm{P}_x)$. Then use Lemma 6.8 to get that $\mathrm{P}^{\phi_0}_x = \mathrm{P}^{(\infty)}_x$. □

7. Two examples

In this section, we specialize the results of the previous sections to the case of the multitype Feller process and of the superdiffusion.

7.1. **The multitype Feller diffusion.**

The multitype Feller diffusion is the superprocess with finite state space: $E = \{1,\dots,K\}$ for K integer. In this case, the spatial motion is a pure jump Markov process, which will be assumed irreducible. Its generator $\mathcal{L}$ is a square matrix $(q_{ij})_{1\le i,j\le K}$ of size K with lines summing up to 0, where q_{ij} gives the transition rate from i to j for $i\neq j$. The functions β and α defining the branching mechanism (2) are vectors of size K: this implies that $(H2)$ and $(H3)$ automatically hold. For more details about the construction of finite state space superprocess, we refer to [14], example 2, p. 10, and to [6] for investigation of the Q-process.

The generalized eigenvalue λ_0 is defined by:

$$\lambda_0 = \sup\{\ell\in\mathbb{R}, \exists u>0 \text{ such that } (\mathrm{Diag}(\beta)-\mathcal{L})u = \ell u\}, \tag{86}$$

where $\mathrm{Diag}(\beta)$ is the diagonal $K\times K$ matrix with diagonal coefficients derived from the vector β. We stress that the generalized eigenvalue is also the Perron Frobenius eigenvalue, *i.e.* the eigenvalue with the maximum real part, which is real by Perron Frobenius theorem, see [30], Exercise 2.11. Moreover, the associated eigenspace is one-dimensional. We will denote by ϕ_0 and $\tilde{\phi}_0$ its generating left, resp. right, eigenvectors, normalized so that $\sum_{i=1}^K \phi_0(i)\tilde{\phi}_0(i) = 1$, and the coordinates of ϕ_0 and $\tilde{\phi}_0$ are positive.

We first check that the two assumptions we made in Section 6 are satisfied.

Lemma 7.1. *Assumptions* $(H8)$ *and* $(H9)$ *hold with* $\pi = \phi_0\, \tilde{\phi}_0$.

Proof. Assumption $(H8)$ is obvious in the finite state space setting. Assumption $(H9)$ is a classical statement about irreducible finite state space Markov Chains. □

Lemma 7.2. *Assume* $\lambda_0 \geq 0$. *Then* $(H1)$, $(H4)$ *and* $(H5)_\nu$ *hold.*

Proof. Assumption $(H2)$ and $(H8)$ hold according to Lemma 7.1. Together with $\lambda_0 \geq 0$, this allows us to apply Lemma 6.2 to obtain $(H1)$. Then use Proposition 6.8 to get $(H4)$ and Corollary 6.9 to get $(H5)_\nu$. □

Lemma 7.3. *Assume* $\lambda_0 > 0$. *Then* $(H6)$ *and* $(H7)$ *holds.*

Proof. We apply Proposition 6.12 to prove $(H6)$ and Lemma 6.5 to prove $(H7)$. □

Recall that $\mathrm{P}_x^{(h)}$ and $\mathrm{P}_x^{(\infty)}$ were defined in (54) and (58) respectively.

Lemma 7.4. *We have:*

(i) $\mathrm{P}_x^{(h)}$ *is a continuous time inhomogeneous Markov chain on* $[0, h)$ *issued from* x *with transition rates from* i *to* j, $i \neq j$, *equal to* $\frac{\partial_h v_{h-t}(j)}{\partial_h v_{h-t}(i)} q_{ij}$ *at time* t, $0 \leq t < h$.

(ii) $\mathrm{P}_x^{(\infty)}$ *is a continuous time homogeneous Markov chain on* $[0, \infty)$ *issued from* x *with transition rates from* i *to* j, $i \neq j$, *equal to* $\frac{\phi_0(j)}{\phi_0(i)} q_{ij}$.

Proof. The first item is a consequence of a small adaptation of Lemma 3.2 for time dependent function. Namely, let $g_t(x)$ be a time dependent function. Consider the law of process (t, Y_t) and consider the probability measure P^g defined by (9) with $g(t, Y_t) = g_t(Y_t)$. Denoting by $\mathcal{L}_t^g$ the generator of (the inhomogeneous Markov process) Y_t under P^g, we have that:

$$\forall u \in \mathcal{D}_g(\mathcal{L}), \quad \mathcal{L}_t^g(u) = \frac{\mathcal{L}(g_t u) - \mathcal{L}(g_t) u}{g_t}. \tag{87}$$

Recall that for all vector u, $\mathcal{L}(u)(i) = \sum_{j \neq i} q_{ij}\big(u(j) - u(i)\big)$. Then apply (87) to the time dependent function $g_t(x) = \partial_t v_{h-t}(x)$, and note that $\mathrm{P}^g = \mathrm{P}^{(h)}$ thanks to (54). For the second item, observe that Proposition 6.8 identifies $\mathrm{P}_x^{(\infty)}$ with P^{ϕ_0}. Use then Lemma 3.2 to conclude. □

William's decomposition under $\mathbb{N}_x^{(h)}$ (Propositions 4.14) together with the convergence of this decomposition (Theorem 5.5) then hold under the assumption $\lambda_0 \geq 0$. Convergence of the distribution of the superprocess near its extinction time under $\mathbb{N}_x^{(h)}$ (Proposition 5.9) holds under the stronger assumption $\lambda_0 > 0$. We were unable to derive an easier formula for $\mathrm{P}^{(-\infty)}$ in this context.

Remark that Lemma 5.1, Definition 5.7 and Corollary 5.8 give a precise meaning to the "interactive immigration" suggested by the authors in Remark 2.8. of [6].

7.2. **The superdiffusion.** The superprocess associated to a diffusion is called superdiffusion. We first define the diffusion and the relevant quantities associated to it, and take for that the general setup from [27]. Here E is an arbitrary domain of $\mathbb{R}^K$ for K integer. Let a_{ij} and b_i be in $\mathcal{C}^{1,\mu}(E)$, the usual Hölder space of order $\mu \in [0,1)$, which consists of functions whose first order derivatives are locally Hölder continuous with exponent μ, for each i,j in $\{1,\dots,K\}$. Moreover, assume that the functions $a_{i,j}$ are such that the matrix $(a_{ij})_{(i,j)\in\{1\dots K\}^2}$ is positive definite. Define now the generator $\mathcal{L}$ of the diffusion to be the elliptic operator:

$$\mathcal{L}(u) = \sum_{i=1}^K b_i\, \partial_{x_i} u + \frac{1}{2} \sum_{i,j=1}^K a_{ij}\, \partial_{x_i,x_j} u.$$

The generalized eigenvalue λ_0 of the operator $\beta - \mathcal{L}$ is defined by:

$$\lambda_0 = \sup\{\ell \in \mathbb{R}, \exists u \in \mathcal{D}(\mathcal{L}), u > 0 \text{ such that } (\beta - \mathcal{L})u = \ell\, u\}\cdot \tag{88}$$

Denoting E the expectation operator associated to the process with generator $\mathcal{L}$, we recall an equivalent probabilistic definition of the generalized eigenvalue λ_0:

$$\lambda_0 = -\sup_{A\subset\mathbb{R}^K} \lim_{t\to\infty} \frac{1}{t} \log \mathrm{E}_x\big[\, \mathrm{e}^{-\int_0^t ds\, \beta(Y_s)}\, \mathbf{1}_{\{\tau_{A^c}>t\}}\big],$$

for any $x \in \mathbb{R}^K$, where $\tau_{A^c} = \inf\{t > 0 : Y(t) \notin A\}$ and the supremum runs over the compactly embedded subsets A of $\mathbb{R}^K$. We assume that the operator $(\beta - \lambda_0) - \mathcal{L}$ is critical in the sense that the space of positive harmonic functions for $(\beta - \lambda_0) - \mathcal{L}$ is one dimensional, generated by ϕ_0. In that case, the space of positive harmonic functions of the adjoint of $(\beta - \lambda_0) - \mathcal{L}$ is also one dimensional, and we denote by $\tilde{\phi}_0$ a generator of this space. We further assume that the operator $(\beta - \lambda_0) - \mathcal{L}$ is **product-critical**, *i.e.* $\int_E dx\, \phi_0(x)\, \tilde{\phi}_0(x) < \infty$, in which case we can normalize the eigenvectors in such a way that $\int_E dx\, \phi_0(x)\, \tilde{\phi}_0(x) = 1$. This assumption (already appearing in [15]) is a rather strong one and implies in particular that P^{ϕ_0} is the law recurrent Markov process, see Lemma 7.5 below.

Concerning the branching mechanism, we will assume, in addition to the conditions stated in section 2, that $\alpha \in \mathcal{C}^4(E)$.

Lemma 7.5. *Assume* $(H8)$. *Assumption* $(H9)$ *holds with* $\pi(dx) = \phi_0(x)\, \tilde{\phi}_0(x)\, dx$.

Proof. We assume that $(\beta - \lambda_0) - \mathcal{L}$ is a critical operator which is product critical. Note that $-\mathcal{L}^{\phi_0}$ is the (usual) h-transform of the operator $(\beta-\lambda_0)-\mathcal{L}$ with $h = \phi_0$, where the h-transform of $\mathcal{L}(\cdot)$ is $\mathcal{L}(h\cdot)/h$. Then Remark 5 of [15] implies that $-\mathcal{L}^{\phi_0}$ is again a critical operator which is also product critical with corresponding ϕ_0 and $\tilde{\phi}_0$ given by 1 and $\phi_0\, \tilde{\phi}_0$. Then Theorem 9.9 p.192 of [27], see (9.14), states that $(H9)$ holds. □

Note that the non negativity of the generalized eigenvalue of the operator $(\beta-\mathcal{L})$ now characterizes in general the local extinction property (the superprocess X suffers local extinction if its restrictions to compact domains of E suffers global extinction); see [16] for more details on this topic. However, under the boundedness assumption we just made on α and ϕ_0, the extinction property $(H1)$ holds, as will be proved (among other things) in the following Lemma.

Lemma 7.6. *Assume* $\lambda_0 \geq 0$ *and* $(H8)$. *Then* $(H1)$*-*$(H4)$ *and* $(H5)_\nu$ *hold. If moreover* $\lambda_0 > 0$, *then* $(H6)$ *and* $(H7)$ *holds.*

Proof. The assumption $\alpha \in \mathcal{C}^4(E)$ ensures that $(H2)$ and $(H3)$ hold. Then the end of the proof is similar to the end of the proof of Lemma 7.2 and the proof of Lemma 7.3. □

Recall that $\mathrm{P}_x^{(h)}$ and $\mathrm{P}_x^{(\infty)}$ were defined in (54) and (58).

Lemma 7.7. *We have:*

- $\mathrm{P}_x^{(h)}$ *is an inhomogeneous diffusion on* $[0, h)$ *issued from* x *with generator at time* $t \in [0, h)$*:* $\left(\mathcal{L} + a\dfrac{\nabla \partial_h v_{h-t}}{\partial_h v_{h-t}} \nabla.\right)$.
- $\mathrm{P}_x^{(\infty)}$ *is an homogeneous diffusion on* $[0, \infty)$ *issued from* x *with generator* $\left(\mathcal{L} + a\frac{\nabla \phi_0}{\phi_0} \nabla.\right)$.

Proof. The proof is similar to the proof of Lemma 7.4. □

William's decomposition under $\mathbb{N}_x^{(h)}$ (Propositions 4.14) together with the convergence of this decomposition (Theorem 5.5) then hold under the assumption $\lambda_0 \geq 0$ and $(H8)$. Convergence of the distribution of the superprocess near its extinction time under $\mathbb{N}_x^{(h)}$ (Proposition 5.9) holds under the stronger assumption $\lambda_0 > 0$.

Remark 7.8. Engländer and Pinsky offer in [17] a decomposition of supercritical non-homogeneous superdiffusion using immigration on the backbone formed by the prolific individuals (as denominated further in Bertoin, Fontbona and Martinez [4]). It is interesting to note that the generator of the backbone is $\mathcal{L}^w$ where w formally satisfies the evolution equation $\mathcal{L}w = \psi(w)$, whereas the generator of the spine of the Q process investigated in Theorem 5.5 is $\mathcal{L}^{\phi_0}$ where ϕ_0 formally satisfies $\mathcal{L}\phi_0 = \beta\phi_0$. In particular, we notice that the generator of the backbone $\mathcal{L}^w$ depends on both β and α and that the generator of our spine $\mathcal{L}^{\phi_0}$ does not depend on α.

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